Aerodynamic Theory

A General Review of Progress

Under a Grant of the Guggenheim Fund
for the Promotion of Aeronautics

William Frederick Durand
Editor-in-Chief

Volume I

Div. A · Mathematical Aids · W. F. Durand
Div. B · Fluid Mechanics, Part I · W. F. Durand
Div. C · Fluid Mechanics, Part II · Max M. Munk
Div. D · Historical Sketch · R. Giacomelli
and E. Pistolesi

With 151 Figures

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GENERAL PREFACE

During the active life of the Guggenheim Fund for the Promotion of Aeronautics, provision was made for the preparation of a series of monographs on the general subject of Aerodynamic Theory. It was recognized that in its highly specialized form, as developed during the past twenty-five years, there was nowhere to be found a fairly comprehensive exposition of this theory, both general and in its more important applications to the problems of aeronautic design. The preparation and publication of a series of monographs on the various phases of this subject seemed, therefore, a timely undertaking, representing, as it is intended to do, a general review of progress during the past quarter century, and thus covering substantially the period since flight in heavier than air machines became an assured fact.

Such a present taking of stock should also be of value and of interest as furnishing a point of departure from which progress during coming decades may be measured.

But the chief purpose held in view in this project has been to provide for the student and for the aeronautic designer a reasonably adequate presentation of background theory. No attempt has been made to cover the domains of design itself or of construction. Important as these are, they lie quite aside from the purpose of the present work.

In order the better to suit the work to this main purpose, the first volume is largely taken up with material dealing with special mathematical topics and with fluid mechanics. The purpose of this material is to furnish, close at hand, brief treatments of special mathematical topics which, as a rule, are not usually included in the curricula of engineering and technical cours s and thus to furnish to the reader, at least some elementary notions of various mathematical methods and resources, of which much use is made in the development of aerodynamic theory. The same material should also be acceptable to many who from long disuse may have lost facility in such methods and who may thus, close at hand, find the means of refreshing the memory regarding these various matters.

The treatment of the subject of Fluid Mechanics has been developed in relatively extended form since the texts usually available to the technical student are lacking in the developments more especially of interest to the student of aerodynamic theory. The more elementary treatment by the General Editor is intended to be read easily by the average technical graduate with some help from the topics comprised in Division A. The more advanced treatment by Dr. Munk will call

for some familiarity with space vector analysis and with more advanced mathematical methods, but will commend itself to more advanced students by the elegance of such methods and by the generality and importance of the results reached through this generalized three-dimensional treatment.

In order to place in its proper setting this entire development during the past quarter century, a historical sketch has been prepared by Professor Giacomelli whose careful and extended researches have resulted in a historical document which will especially interest and commend itself to the study of all those who are interested in the story of the gradual evolution of the ideas which have finally culminated in the developments which furnish the main material for the present work.

The remaining volumes of the work are intended to include the general subjects of: The aerodynamics of perfect fluids; The modifications due to viscosity and compressibility; Experiment and research, equipment and methods; Applied airfoil theory with analysis and discussion of the most important experimental results; The non-lifting system of the airplane; The air propeller; Influence of the propeller on the remainder of the structure; The dynamics of the airplane; Performance, prediction and analysis; General view of airplane as comprising four interacting and related systems; Airships, aerodynamics and performance; Hydrodynamics of boats and floats; and the Aerodynamics of cooling.

Individual reference will be made to these various divisions of the work, each in its place, and they need not, therefore, be referred to in detail at this point.

Certain general features of the work editorially may be noted as follows:

1. Symbols. No attempt has been made to maintain, in the treatment of the various Divisions and topics, an absolutely uniform system of notation. This was found to be quite impracticable.

Notation, to a large extent, is peculiar to the special subject under treatment and must be adjusted thereto. Furthermore, beyond a few symbols, there is no generally accepted system of notation even in any one country. For the few important items covered by the recommendations of the National Advisory Committee for Aeronautics, symbols have been employed accordingly. Otherwise, each author has developed his system of symbols in accordance with his peculiar needs.

At the head of each Division, however, will be found a table giving the most frequently employed symbols with their meaning. Symbols in general are explained or defined when first introduced.

2. General Plan of Construction. The work as a whole is made up of *Divisions*, each one dealing with a special topic or phase of the general

subject. These are designated by letters of the alphabet in accordance with the table on a following page.

The Divisions are then divided into chapters and the chapters into sections and occasionally subsections. The Chapters are designated by Roman numerals and the Sections by numbers in bold face.

The Chapter is made the unit for the numbering of sections and the section for the numbering of equations. The latter are given a double number in parenthesis, thus (13.6) of which the number at the left of the point designates the section and that on the right the serial number of the equation in that section.

Each page carries at the top, the chapter and section numbers.

W. F. Durand

Stanford University, California
January, 1934

GENERAL LIST OF DIVISIONS WITH AUTHORS

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A. Mathematical Aids

W. F. Durand — Professor (Emeritus) of Mechanical Engineering, Stanford University, Calif., Member of the National Advisory Committee for Aeronautics.

B. Fluid Mechanics, Part I

W. F. Durand

C. Fluid Mechanics, Part II

Max M. Munk — Lecturer in Aerodynamics at the Catholic University of America, Washington, D. C., and Technical Editor of the "Aero Digest".

D. Historical Sketch

R. Giacomelli — Lecturer in History of Mechanics at the University of Rome, Italy, and Editor of "L'Aerotecnica".

with the collaboration of

E. Pistolesi — Professor of Mechanics at the Royal School of Engineering at Pisa, Italy, and Editor-in-Chief of "L'Aerotecnica".

Volume II.

E. General Aerodynamic Theory—Perfect Fluids

Th. von Kármán — Director of the Guggenheim Aeronautics Laboratory, California Institute of Technology, Pasadena, Calif., and formerly Director of the Aerodynamic Institute, Aachen, Germany.

J. M. Burgers — Professor of Aero- and Hydrodynamics at the Technische Hoogeschool at Delft, Holland.

Volume III.

F. Application of a Discontinuous Potential to the Theory of Lift with Single Burbling

C. Witoszyński — Professor of Aerodynamics at the Warsaw Polytechnial School and Director of the Warsaw Aerodynamic Institute, Poland.

M. J. Thompson — Assistant Professor of Aeronautical Engineering at the University of Michigan, Ann Arbor, Mich.

G. The Mechanics of Viscous Fluids

L. Prandtl — Professor in Applied Mechanics at the University of Göttingen, Germany, and Director of the Kaiser Wilhelm Institute for Fluid Research.

H. The Mechanics of Compressible Fluids

G. I. Taylor — Yarrow Research Professor of the Royal Society, Fellow of Trinity College, Cambridge, England.

J. W. Maccoll — Research Officer, Department of External Ballistics, Ordnance Committee, Woolwich, England.

I. Experimental Research in Aerodynamics—Equipment and Methods

A. Toussaint — Director of the Aerodynamic Laboratory, Saint-Cyr-l'École, France.

E. Jacobs — Associate Aeronautical Engineer, in charge of the National Advisory Committee for Aeronautics' variable-density wind tunnel, Langley Field, Virginia.

Volume IV.

J. Applied Airfoil Theory

A. Betz — Professor at the University and Director of the Aerodynamic Research Institute at Göttingen, Germany.

K. Aerodynamics of the Airplane Body (Non-Lifting System) Drag and Influence on Lifting System

C. Wieselsberger — Professor of Aerodynamics and Director of the Aerodynamic Institute, Technische Hochschule, Aachen, Germany.

L. Airplane Propellers

H. Glauert — Past Fellow of Trinity College, Cambridge, England; Principal Scientific Officer at the Royal Aircraft Establishment, Farnborough.

M. Influence of the Propeller on other Parts of the Airplane Structure

C. Koning — Rijks-Studiedienst voor de Luchtvaart, Amsterdam, Holland.

Volume V.

N. Dynamics of the Airplane

B. Melvill Jones — Professor of Aeronautical Engineering in the University of Cambridge, England, Member of the Aeronautical Research Committee of Great Britain.

O. Performance of Airplanes

L. V. Kerber — Former Chief Aerodynamics Branch Materiel Division, U. S. Army Air Corps, and former Chief, Engineering Section Aeronautics Branch, Department of Commerce.

Volume VI.

P. Airplane as a Whole—General View of Mutual Interactions Among Constituent Systems

M. Panetti — Professor of Applied Mechanics and Director of the School of Aeronautics in the R. Politecnico di Torino, Italy.

Q. Aerodynamic Theory of Airships

Max M. Munk — Lecturer in Aerodynamics at the Catholic University of America, Washington, D.C., and Technical Editor of the "Aero Digest".

R. Performance of Airships

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W. Klemperer — Research Engineer of the Goodyear Zeppelin Company, Akron, Ohio.

S. Hydrodynamics of Boats and Floats

E. G. Barrillon — Director of the Naval Experimental Tank, Paris, France.

T. Aerodynamics of Cooling

H. L. Dryden — Physicist in the United States Bureau of Standards, Chief of the Aerodynamics Section, Washington, D.C.

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By **W. F. Durand,**

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FLUID MECHANICS, PART I

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DIVISION C

FLUID MECHANICS, PART II

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DIVISION D

HISTORICAL SKETCH

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with the collaboration of
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NOTATION

The following table comprises a list of the principal notations employed in the present Volume. Notations not listed are either so well understood as to render mention unnecessary, or are only rarely employed and are explained as introduced. Where occasionally a symbol is employed with more than one meaning, the local context will make the significance clear.

DIVISION A

z	The complex quantity $(x + i\,y)$: The vector $(x + i\,y)$
R	Radius of curvature, VI 8
r	Radius vector
S	Surface or area
θ	Any angle, usually the vector angle with X
u	Velocity in the direction of the axis X
v	Velocity in the direction of the axis Y
w	Velocity in the direction of the axis Z
w	Any function of z
V	Usually resultant or total velocity
n	Velocity along radius vector
c	Velocity $\perp$ to radius vector
∇^2	Laplacian, I 2 VII 3
φ	The scalar part of w. Also *potential*
ψ	The imaginary part of w
Π	A non-dimensional function IV 3
μ	Coefficient of viscosity, IV 4
ν	Coefficient of kinematic viscosity IV 4
L	Line integral, VI 3
ϱ	Density

DIVISION B

X, Y, Z	Coordinate axes
x, y, z	Coordinates along X, Y, Z
z	The complex quantity $(x + i\,y)$
r	Radius vector
θ	Angle, usually vector angle with X
u, U	Velocity in the direction of X
v	Velocity in the direction of Y
w	Velocity in the direction of Z, Induced velocity, III 2
w	Potential function $= (\varphi + i\,\psi)$
V	Resultant or total velocity
u', v'	Velocities in the directions of X and Y with axes fixed relative to the indefinite mass of fluid, VII 1
n	Component velocity in the direction of r
c	Component velocity $\perp$ to the direction of r
ω	Angular velocity
Γ, γ	Circulation or vorticity, III 1
$\triangle$	Difference
∇^2	Laplacian, A VII 3

φ	Potential
ψ	Stream function
φ', ψ', w'	Potential, Stream function and potential function with axes fixed relative to the indefinite mass of fluid, VII 3
p, P	Pressure
E	Energy
M	Moment or momentum, also $= 2a\mu$, IV 10
k	Used for $\Gamma/2\pi$, III 3
m	Strength of source or sink, II 7
μ	Used for $m/2\pi$
N	Line or direction along a normal
ϱ	Density
t	Time

DIVISION C

X, Y, Z	Coordinate axes
x, y, z	Coordinates along X, Y, Z
$\bar{y}$	Distance ⊥ to X along an axial plane, IV 4
μ, ζ	Semi-elliptic coordinates, VII 2
λ, μ, ν	Elliptic coordinates, VIII 2
r	Radius vector
R	Radius
S	Surface or area
θ	Angle, usually the vector angle with X
u, v, w	Component velocities along X, Y, Z
V	Velocity, usually total velocity
Ω, ω	Angular velocity, axial angle, IV 4
Γ	Circulation, vorticity
∇	The Operator "Del", I (2.3)
∇^2	Laplacian, II 3
φ	Potential
ψ	Stream function
p	Pressure
T	Energy
m	Strength of Source, IV 1
M	Strength of Doublet, IV 3
E, F, K	Elliptic integrals, VIII 5
n	Direction along a normal
ϱ	Density
t	Time

DIVISION A

MATHEMATICAL AIDS

By

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PREFACE

Reference has been already made in the general preface to the purpose of this Division of the work as a whole. The choice of topics has been such as will promise the most immediate aid to those whose mathematical preparation may not have included all the methods conveniently and usefully employed in the development of various phases of aerodynamic theory. In all cases the treatment has, of necessity, been partial and confined to the more elementary phases of the subject. No attempt has been made to include the subject of differential equations, this for the two reasons: first, the very considerable space which would be required for a treatment covering in any reasonable degree the use of this mathematical discipline in its many applications to the various problems arising in aerodynamic theory and second, the growing extent to which this subject is now found in engineering and technical curricula and the many excellent text books which are available for the interested reader.

The subjects of elliptic integrals and of elliptical functions has also been omitted by reason of lack of space for any reasonably adequate treatment and for the further reason that readers interested in those phases of the subject in which this discipline would be of special significance would, presumably, already be acquainted with them.

CHAPTER I

THE COMPLEX VARIABLE (x + i y)

1. Introductory. In the present section we are concerned primarily with the symbol i as the equivalent of $\sqrt{-1}$ and with the mathematical results which follow from the use of the complex form $(x + i\,y)$ considered as a single variable z, and as such, made the subject of various functional operations. Thus, we put

$$z = x + i\,y, \text{ and then write} \tag{1.1}$$

$$w = f(z) \tag{1.2}$$

This means that the *complex variable* $(x + iy)$ is to be made the subject of a development in functional form and thus subject to examination in the same manner as any simple function $f(x)$.

Thus as an illustration let

$$w = z^2 = (x + iy)^2. \text{ Expanding we have}$$
$$w = (x^2 - y^2) + 2i\,xy$$

The developed form is thus seen to consist of two parts, one real, $(x^2 - y^2)$ and the other, $2i\,xy$, containing the imaginary i. This form of the development of a function of $(x + iy)$ is typical and general. The development will consist of a real part (*scalar*) and of a part affected by the factor i and these two parts may be viewed analytically simply as the *real* and *imaginary* parts of the development. To designate these we adopt the notation

$$\varphi = \text{scalar or real part}$$
$$\psi = \text{imaginary part}$$

We then have
$$w = \varphi + i\psi \tag{1.3}$$

It will be noted that the two functions φ and ψ cannot arise separately. They always appear together, one as the correlative or complement of the other, and the peculiar properties and relations of these two functions will be found to have important applications in the study of the problems of fluid mechanics.

2. Properties of the Functions φ and ψ. From the equation

$$z = x + iy \text{ we have}$$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 1 & \qquad \frac{\partial x}{\partial z} &= 1 \\ \frac{\partial z}{\partial y} &= i & \qquad \frac{\partial y}{\partial z} &= -i \end{aligned}\right\} \tag{2.1}$$

Then taking
$$w = f(z) = f(x + iy) = \varphi + i\psi \tag{2.2}$$

we have
$$\left.\begin{aligned} \frac{dw}{dz} &= \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial z} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} \\ \frac{dw}{dz} &= \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial z} = -i\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial y} \end{aligned}\right\} \tag{2.3}$$

Then from (2.1) and (2.3)

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \frac{dw}{dz}\,\frac{\partial z}{\partial x} = \frac{dw}{dz} \\ \frac{\partial w}{\partial y} &= \frac{dw}{dz}\,\frac{\partial z}{\partial y} = i\frac{dw}{dz} \end{aligned}\right\} \tag{2.4}$$

Whence by comparison of these two equations:

$$\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y} \text{ and } \frac{\partial w}{\partial y} = i\frac{\partial w}{\partial x} \tag{2.5}$$

Again in (2.3) equating real to real and imaginary to imaginary:

$$\left.\begin{aligned}\frac{\partial \varphi}{\partial x} &= \frac{\partial \psi}{\partial y}\\ \frac{\partial \varphi}{\partial y} &= -\frac{\partial \psi}{\partial x}\end{aligned}\right\} \tag{2.6}$$

These relations are of great importance and lie at the foundation of the special significance of the functions φ and ψ in the problems of fluid mechanics.

Again from (2.3) and (2.6)

$$\left.\begin{aligned}\frac{dw}{dz} &= \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}\\ \frac{dw}{dz} &= \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}\end{aligned}\right\} \tag{2.7}$$

Attention may be called to (2.7) as having important applications in the later development of the subject of fluid mechanics.

Again from the pair of derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ we have:

$$\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial^2 \varphi}{\partial y\,\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial^2 \varphi}{\partial x\,\partial y}$$

But if φ is a function expressible in terms of x and y, we know that

$$\frac{\partial^2 \varphi}{\partial x\,\partial y} = \frac{\partial^2 \varphi}{\partial y\,\partial x} \tag{2.8}$$

That is, the order in which the partial derivatives are taken is indifferent. This is furthermore readily verified by taking the partial derivatives of any function of x and y, first with reference to x and then y and then in the inverse order. Hence,

$$\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = 0 \tag{2.9}$$

And similarly,

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = 0 \tag{2.10}$$

Now going back to (2.6), take partial derivatives in the first with respect to x and in the second with respect to y.

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y\,\partial x}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial x\,\partial y}$$

But as above, the expressions on the right are equal, and with opposite signs. Hence,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{2.11}$$

In a similar manner by taking partial derivatives in the inverse order, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2.12}$$

Again from (2.1)
$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial z^2}\left(\frac{\partial z}{\partial x}\right)^2 = \frac{\partial^2 w}{\partial z^2}$$
$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2 = -\frac{\partial^2 w}{\partial z^2}$$
or
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{2.13}$$

This result might have been foreseen as a consequence of the relation $w = \varphi + i\psi$.

As an operator, the expression on the left in (2.11), (2.12), (2.13) is sometimes called the Laplacian of the function (φ, ψ, or w) and is indicated by the symbol ∇^2. Thus

$$\nabla^2\varphi = 0,\ \nabla^2\psi = 0,\ \nabla^2 w = 0$$

The physical meaning of this symbol will appear in VII 3.

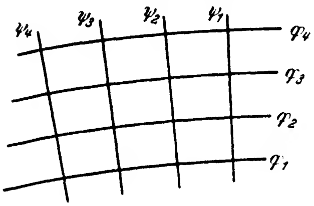

Fig. 1.

It thus appears in general that the development of the complex variable, $z = (x + iy)$, as a function, $w = f(z)$, will always lead to the two correlative functions φ and ψ with mutual relations as expressed in (2.6) and each with the characteristic property as expressed in (2.11) and (2.12).

It will be noted that the property expressed in (2.8) is common to any and all functions expressed in terms of x and y while the property expressed in (2.11) or (2.12) is not general, but is peculiar as a distinguishing characteristic of the two functions, φ and ψ, which result from the expanded form of a function of the complex variable $(x + iy)$.

Next, suppose, as in Fig. 1 a series of curves determined by the equation $\varphi = \varphi_1, \varphi_2, \varphi_3$, etc. and a second series determined by the equations $\psi = \psi_1, \psi_2, \psi_3$, etc. and assume any one of one series to intersect any one of the other. Then at the point of intersection, apply (2.6) and divide one of these by the other. This will give

$$\left.\frac{dy}{dx}\right]_\varphi = -\left.\frac{dx}{dy}\right]_\psi \tag{2.14}$$

But dy/dx is the tangent of the angle of slope to the axis of X and this equation shows that one such tangent is the negative reciprocal of the other and hence that the curves are at right angles to each other at the point of intersection. This will be true for every such intersection and the two sets of such curves will therefore always have their mutual intersections at 90^0. One set of curves is therefore orthogonal to the other. We thus reach the interesting result that if $(x + iy)$ be expanded into any form of algebraic function, giving the two correlative functions φ

and ψ as above, and if then $\varphi = \varphi_1, \varphi_2, \varphi_3$, etc. and $\psi = \psi_1, \psi_2, \psi_3$, etc. be taken as equations to two sets or families of curves, such curves will form an orthogonal system.

Illustration: $w = z^2 = (x + i\,y)^2 = x^2 + 2\,i\,x\,y - y^2$

Hence
$$\varphi = x^2 - y^2$$
$$\psi = 2\,x\,y$$

The series of curves for $\varphi = \varphi_1, \varphi_2$, etc. will be a series of hyperbolas symmetrical about X and Y as shown in Fig. 2 while the series $\psi = \psi_1, \psi_2$, etc. will be a series of rectangular hyperbolas lying in the first and third quadrants as shown. These two sets will then always intersect at an angle of 90^0.

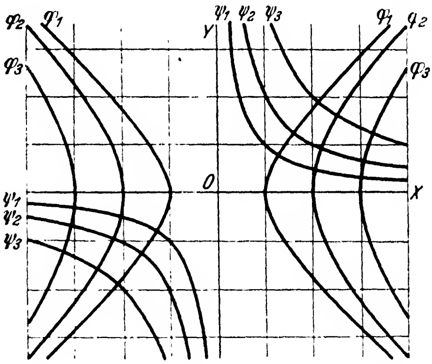

Fig. 2.

3. The Inverse Relation z = F (w). If w is a function of z, it follows inversely that z will be a function of w; and it is sometimes advantageous to make use of this inverse relation. Similar to 2 we shall then have

$$z = F(w) = F(\varphi + i\,\psi) = x + i\,y$$

$$\left.\begin{aligned}
&\frac{\partial w}{\partial \varphi} = 1 \qquad \frac{\partial \varphi}{\partial w} = 1 \\
&\frac{\partial w}{\partial \psi} = i \qquad \frac{\partial \psi}{\partial w} = -i \\
&\frac{dz}{dw} = \frac{\partial z}{\partial \varphi}\frac{\partial \varphi}{\partial w} = \frac{\partial x}{\partial \varphi} + i\frac{\partial y}{\partial \varphi} \\
&\frac{dz}{dw} = \frac{\partial z}{\partial \psi}\frac{\partial \psi}{\partial w} = -i\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \psi}
\end{aligned}\right| \tag{3.1}$$

and
$$\frac{\partial z}{\partial \varphi} = \frac{\partial z}{\partial w} \qquad \frac{\partial z}{\partial \psi} = i\frac{\partial z}{\partial w}$$
$$\frac{\partial z}{\partial \varphi} = -i\frac{\partial z}{\partial \psi} \qquad \frac{\partial z}{\partial \psi} = i\frac{\partial z}{\partial \varphi}$$

Then parallel to (2.6)
$$\left.\begin{aligned}
\frac{\partial x}{\partial \varphi} &= \frac{\partial y}{\partial \psi} \\
\frac{\partial y}{\partial \varphi} &= -\frac{\partial x}{\partial \psi}
\end{aligned}\right| \tag{3.2}$$

And again parallel to (2.7)
$$\left.\begin{aligned}
\frac{dz}{dw} &= \frac{\partial x}{\partial \varphi} - i\frac{\partial x}{\partial \psi} \\
\frac{dz}{dw} &= \frac{\partial y}{\partial \psi} + i\frac{\partial y}{\partial \varphi}
\end{aligned}\right| \tag{3.3}$$

In the same manner as for $w = f(z)$ we may show also that the two sets of curves, derived from putting $\varphi = const.$ and $\psi = const.$ will intersect always at right angles and will, therefore, form an orthogonal field.

By way of illustration, assume

$$z = c\cos w = c\cos(\varphi + i\psi). \quad \text{Whence (see 11)}$$
$$x = c\cos\varphi\cosh\psi$$
$$y = -c\sin\varphi\sinh\psi. \quad \text{Whence}$$
$$\frac{x^2}{c^2\cosh^2\psi} + \frac{y^2}{c^2\sinh^2\psi} = 1$$
$$\frac{x^2}{c^2\cos^2\varphi} - \frac{y^2}{c^2\sin^2\varphi} = 1 \quad \text{(see 10)}$$

The curves for $\psi = const.$ are ellipses and those for $\varphi = const.$ are hyperbolas. These conics will have common foci at $\pm(c, 0)$ and intersect always at right angles as shown in Division B Fig. 52.

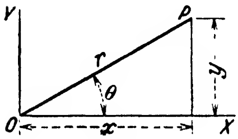

Fig. 3.

4. The Complex x + iy as the Location of a Point in a Plane. Anticipating in some small degree the application of the complex variable to the treatment of vectors and to the subject of vector algebra, we next proceed with the development of certain important relations growing out of the use of the complex $x + iy$ as the designation of a point in a plane. Thus in Fig. 3 we take x as the abscissa and y as the ordinate and thus locate a point P in the plane XY. In terms of polar coordinates, the point might also be located by the length r laid off at the angle θ where

$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\frac{y}{x}$$

If then we take the equation $z = x + iy$ and substitute for x and y in terms of r and θ as in Fig. 3 we shall have

$$z = r(\cos\theta + i\sin\theta) \tag{4.1}$$

Now the application of Maclaurin's theorem to the expansion of e^θ gives: $\quad e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \cdots$

If instead of θ we put $i\theta$, we may write the result in the form:

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots\right]$$

But the expression within the first bracket is known to be the expansion of $\cos\theta$, and similarly that in the second bracket is the expansion of $\sin\theta$, both in ascending powers of θ. Hence as an algebraic identity we may write: $\quad e^{i\theta} = \cos\theta + i\sin\theta \qquad (4.2)$

This is to be understood simply as a statement of the equivalence of these two algebraic forms, or of their identity when expanded in terms of θ.

Comparing (4.1) and (4.2) it is clear that we may write the complex variable $z = x + iy$ in the form

$$z = re^{i\theta} \tag{4.3}$$

and in whatever way $z = (x + iy)$ may be used as the representation of a point in a plane, or as a vector or otherwise, we may equally well employ the expression $re^{i\theta}$. It should be clearly noted that at this point we are only concerned with the establishment of the analytical equality of the three expressions:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{4.4}$$

all under the conventions expressed in Fig. 3.

5. Results Growing Out of the Expression of the Complex Variable in the Exponential and Circular Function Forms. The series of equalities $z = (x + iy) = r(\cos\theta + i\sin\theta) = re^{i\theta}$ leads to many interesting and important relations. Among these a few are selected as of major importance for our present purpose. Most of these are self evident, or follow directly from the preceding sections.

$$(\cos\theta + i\sin\theta)^m = (e^{i\theta})^m = e^{im\theta}$$

whence $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ (a)

This is the well known de Moivre's theorem.

$$\frac{z}{r} = \cos\theta + i\sin\theta = e^{i\theta} \tag{b}$$

$$\frac{r}{z} = \cos\theta - i\sin\theta = e^{-i\theta} \tag{c}$$

Putting $-\theta$ for θ in (b) gives the last two expressions in (c). But $e^{-i\theta}$ is the reciprocal of $e^{i\theta}$. Hence the first term must be the reciprocal of z/r or r/z. Then from (b) and (c)

$$2\cos\theta = \frac{z}{r} + \frac{r}{z} = e^{i\theta} + e^{-i\theta} \tag{d}$$

$$2i\sin\theta = \frac{z}{r} - \frac{r}{z} = e^{i\theta} - e^{-i\theta} \tag{e}$$

Whence
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{f}$$

$$i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{g}$$

Again from (b) and (c)

$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1 \tag{h}$$

This is readily verified by direct multiplication.

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{r}e^{-i\theta} \tag{i}$$

6. The Integration of Functions of a Complex Variable. The complex variable z may appear in integral expressions the same as a real variable. Thus we may be concerned with integrals of the general form:

$$\int f(z)\, dz \tag{6.1}$$

If $f(z)$ is expanded, it will take the form $u + iv$ where u and v are real functions of x and y. The expanded integral will thus take the form:

$$\int (u + iv)(dx + i dy) = \int (u dx - v dy) + i \int (v dx + u dy) \tag{6.2}$$

The integral is thus reduced to the form of four integrals, each involving real functions u and v.

It now becomes a matter of interest to inquire under what conditions the values of the two integrals in (6.2), when the path of integration is carried around a closed contour, will become zero.

Assume the existence of a function φ of x and y and put

$$P = \frac{\partial \varphi}{\partial x} \text{ and } Q = \frac{\partial \varphi}{\partial y} \tag{6.3}$$

Then we shall have

$$\int_A^B (P dx + Q dy) = \int \left(\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \right) = \varphi \Big]_A^B \tag{6.4}$$

where A and B denote the two points between which the integration is carried. Now if the function φ is real, single valued and without singularities (the effect of which will be considered later), it is clear that if the integration is carried around a closed path, the point B comes to A, the two values of φ become the same and the value of the integral becomes zero. The condition that an integral of the form (6.4) shall vanish when the integration is carried around a closed path is therefore the existence of a function φ of the character indicated. But for any such function it is known that we must have

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$$

And from (6.3) this is equivalent to

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{6.5}$$

If we then consider the two integrals of (6.2) respectively parallel to (6.4) and apply to them the condition of (6.5), we shall have the conditions

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \tag{6.6}$$

These are sometimes known as the Cauchy-Riemann equations and are fulfilled when u and v are the two conjugate parts of $f(z)$ as assumed above; that is where we have

$$f(z) = u + iv$$

See also **1**, **2**.

So far as formal integration is concerned we may integrate Eq. (6.1) directly in terms of z, thus giving a result, $F(z)$, which if expanded would take the form $F(z) = U + iV$

We should then have from (6.2)

$$\frac{\partial U}{\partial x} = u, \quad \frac{\partial U}{\partial y} = -v$$
$$\frac{\partial V}{\partial x} = v, \quad \frac{\partial V}{\partial y} = u$$

These again fulfil the Cauchy-Riemann equations and thus show that U and V, related in this way to (6.2) are the conjugate parts of the expansion of some function, $F(z)$.

7. Influence of Singularities. A singular point with reference to a particular function $f(z)$, may, for present purposes, be defined as a point at which the derivative $f'(z)$ has no definite or finite value, no matter from what direction the point is approached. Thus the function $1/z$ has a singular point at the origin and similarly the function $1/(z-z_0)$ has a singular point at $z = z_0$.

Now from the preceding section it appears that, in general, the value of $\int f(z)\,dz$ taken around a closed path will vanish. If, however, this path should inclose a singular point, this result no longer holds.

Take for example the function

$$I = \int_{z_1}^{z_2} \frac{dz}{z} \tag{7.1}$$

for which the origin is a singular point. Integrating we have

$$I = \log z \Big]_{z_1}^{z_2}$$

If now we express z in the form $z = re^{i\theta}$ the integral takes the form

$$\log (re^{i\theta}) \Big]_1^2 = \log \frac{r_2}{r_1} + i\,\theta \Big]_{\theta_1}^{\theta_2}$$

If then a complete circuit is made around a closed path, the value becomes

$$I = \log \frac{r_1}{r_1} + i\,\theta \Big]_0^{2\pi} = 2\pi i \tag{7.2}$$

If instead of integrating directly, we put $z = re^{i\theta}$ in (7.1) the same result develops.

Take again the integral $I = \oint \frac{dz}{z - z_0}$ [1]

around a closed path containing the point z_0. It is readily seen that this case is similar to the preceding except that the singular point is not at the origin.

[1] The symbol $\oint$ implies a line integration carried around a closed contour.

Integrating as before we have $I = \log(z - z_0)$
But here we may write $z = z_0 + re^{i\theta}$ or $z - z_0 = re^{i\theta}$
Hence $I = \log(re^{i\theta}) = 2\pi i$ as before.

8. Cauchy's Theorem. We have seen that the value of the integral $\int f(z)\,dz$ carried around a closed path will vanish if the path does not inclose a singular point, but will not vanish if a singular point is within the circuit. Suppose now that we have a field as in Fig. 4 in which there are two singular points P and Q with a closed path $A\,B\,C\,D\,A$ inclosing both of these points. The integral of $f(z)\,dz$ around this path will not then be zero. Next assume the points P and Q isolated from the remainder of the region by closed paths $a\,b\,c$ and $d\,e\,f$. If then we trace the path $A\,B\,C\,f\,e\,d\,C\,D\,A\,c\,b\,a\,A$, that is in such manner that the region within $A\,B\,C\,D$ and without $a\,b\,c$ and $d\,e\,f$ always lies on the left, the path will not inclose the points P and Q. We may, if we wish, assume an infinitesimal separation between $A\,a$ and $c\,A$ and the same for $C\,d$ and $f\,C$, or otherwise, we may assume $A\,a$ and $c\,A$ to lie, one on one side of the line and the other on the other side, and the same for $C\,d$ and $f\,C$. In either case, identity of the two paths stands simply as the limit toward which these conditions approach.

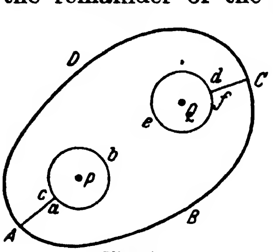

Fig. 4.

For such a path, then, the integral will have zero as its value. But over such a path the values for $A\,a$ and $c\,A$ will cancel and the same for $C\,d$ and $f\,C$. We may then write

$$\oint_{ABCDA} + \oint_{fed} + \oint_{cba} = 0 \tag{8.1}$$

or

$$\oint_{ABCDA} = \oint_{def} + \oint_{abc} \tag{8.2}$$

This important result is known as *Cauchy's Theorem*. It will be noted that in the form (8.1) the cyclical direction followed in traversing the paths about P and Q is the opposite of that for the outer path $A\,B\,C\,D\,A$; while in (8.2) the cyclical directions are all the same. This general result will evidently hold for any number of singular points within the outer closed boundary. If there is but one singular point, we have simply the equality of values of two integrals about a point such as P, the two paths not being the same.

For any point such as P, the expression

$$\frac{1}{2\pi i}\oint f(z)\,dz \tag{8.3}$$

carried around a closed path about P is called *by Cauchy* the ***residue*** of $f(z)$ at the point P.

9. Cauchy's Integral Formula. Consider the integral

$$I = \oint \frac{f(z)\,dz}{z - z_0} \tag{9.1}$$

carried around a closed contour inclosing the point z_0. Take again the companion integral where $f(z)$ becomes $f(z_0)$ and substract the two. This gives

$$\triangle I = \oint \frac{f(z) - f(z_0)}{z - z_0}\,dz \tag{9.2}$$

But $f(z) - f(z_0)$, assuming the function expressible in algebraic form, is readily seen to be exactly divisible by $(z - z_0)$ and the result will be a polynomial in z and z_0. This will have no singularity at the point z_0 and the integral of (9.2) will therefore be zero. This gives us, see 7

$$\oint \frac{f(z)\,dz}{z - z_0} = \oint \frac{f(z_0)\,dz}{z - z_0} = f(z_0) \oint \frac{dz}{z - z_0} = 2\pi i f(z_0) \tag{9.3}$$

This shows then that the value of any integral of the form (9.1) carried around a closed path inclosing a singular point at z_0 is given simply by $2\pi i$ times $f(z_0)$. In this case, $f(z_0)$ is Cauchy's residue for the point z_0.

For a circle of indefinitely small radius r surrounding the point, this result may be otherwise reached as follows. For such a circle, any point on the circumference is given by $z = z_0 + re^{i\theta}$ [see V **11** (f)]. The expression $f(z)$ becomes at the limit $f(z_0)$ and the integral thus becomes

$$I = f(z_0) \oint \frac{dz}{z - z_0}$$

But the integral form, as we have already seen, is equal to $2\pi i$. Hence we have $I = 2\pi i f(z_0)$ as before.

Since, moreover, as shown in **6**, the integral around any one singular point is the same regardless of the path, this same value will hold for any other path and the result has, therefore, the same generality as in (9.3).

It should be noted that the relation

$$\oint \frac{dz}{z - z_0} = 2\pi i$$

only holds for expressions in this exact form. In particular it does not hold for expressions in the more general form.

$$I = \oint \frac{dz}{f(z) - f(z_0)} \tag{9.4}$$

For any such form, the substitution $z = z_0 + re^{i\theta}$, is to be made, retaining only the lowest powers of r, followed by reduction and integration.

As an illustration take the form

$$I = \oint \frac{dz}{z^m - z_0^m}$$

Making the substitution, retaining only the first power of r and reducing we find

$$I = \frac{1}{m z_0^{m-1}} \oint i\,d\theta = \frac{1}{m z_0^{m-1}} (2\pi i)$$

10. Hyperbolic Functions. In relations 5 (f), (g), put $i\,x$ for θ. This will give,

$$\cos(i\,x) = \frac{e^x + e^{-x}}{2}$$

$$\sin(i\,x) = i\,\frac{e^x - e^{-x}}{2} \text{ whence}$$

$$\tan(i\,x) = i\,\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

It will be noted that $\cos(i\,x)$ is real while $\sin(i\,x)$ and $\tan(i\,x)$ are imaginary since they have the factor i. The expression for $\cos(i\,x)$ and the real part of the expressions for $\sin(i\,x)$ and $\tan(i\,x)$ are known as the *hyperbolic cosine*, *hyperbolic sine* and *hyperbolic tangent* of x.

They are written:

$$\cosh x, \ \sinh x \text{ and } \tanh x$$

Thus

$$\left.\begin{aligned} \sin(i\,x) &= i\sinh x \\ \cos(i\,x) &= \cosh x \\ \tan(i\,x) &= i\tanh x \end{aligned}\right\} \tag{10.1}$$

Then

$$\left.\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}\right\} \tag{10.2}$$

and their values may be computed from these exponential expressions.

The reciprocal functions are defined as with the common circular functions:

$$1/\sinh x = \operatorname{cosech} x$$
$$1/\cosh x = \operatorname{sech} x$$
$$1/\tanh x = \coth x$$

The following relations are readily established

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

By combination of the values of $\sinh x$ and $\cosh x$ we have

$$\left.\begin{aligned} e^x &= \cosh x + \sinh x \\ e^{-x} &= \cosh x - \sinh x \end{aligned}\right\} \tag{10.3}$$

and by dividing one of these by the other,

$$\left.\begin{aligned} e^x &= \sqrt{\frac{1 + \tanh x}{1 - \tanh x}} \\ e^{-x} &= \sqrt{\frac{1 - \tanh x}{1 + \tanh x}} \end{aligned}\right\} \tag{10.4}$$

11. Hyperbolic Functions of Imaginaries and Complexes. In the expression $$sinh\,x = \frac{e^x - e^{-x}}{2} \text{ put } i\,x \text{ for } x$$

We shall then have $$sinh\,i\,x = \frac{e^{i\,x} - e^{-i\,x}}{2}$$

Then referring to 5 this becomes

$$sinh\,i\,x = i\,sin\,x$$

Similarly we find
$$cosh\,i\,x = cos\,x$$
$$tanh\,i\,x = i\,tan\,x$$

Again we have $$sinh\,(x + i\,y) = \frac{e^{x+iy} - e^{-(x+iy)}}{2}$$

$$e^{x+iy} = e^x\,(cos\,y + i\,sin\,y)$$

$$e^{-(x+iy)} = e^{-x}\,(cos\,y - i\,sin\,y) \text{ [see 5 (c)] Whence}$$

$$sinh\,(x + i\,y) = sinh\,x\,cos\,y + i\,cosh\,x\,sin\,y$$

In a similar manner we find

$$cosh\,(x + i\,y) = cosh\,x\,cos\,y + i\,sinh\,x\,sin\,y$$

$$tanh\,(x + i\,y) = \frac{tanh\,x + i\,tan\,y}{1 + i\,tanh\,x\,tan\,y}$$

Again take the usual formulas for $sin\,(x \pm y)$ and $cos\,(x \pm y)$ and for y put $i\,y$. Then putting for $sin\,i\,y$ and $cos\,i\,y$ as in (10.1) we have

$$sin\,(x \pm i\,y) = sin\,x\,cosh\,y \pm i\,cos\,x\,sinh\,y$$
$$cos\,(x \pm i\,y) = cos\,x\,cosh\,y \mp i\,sin\,x\,sinh\,y$$

Whence $$tan\,(x \pm i\,y) = \frac{tan\,x \pm i\,tanh\,y}{1 \mp i\,tan\,x\,tanh\,y}$$

12. Inverse Relations. Let
$$u_1 = sinh\,x$$
$$u_2 = cosh\,x$$
$$u_3 = tanh\,x$$

Then keeping everything in terms of u_1 we have

$$u_1^2 = sinh^2\,x$$
$$u_1^2 + 1 = cosh^2\,x \text{ and from (10.3)}$$
$$x = log\,(cosh\,x + sinh\,x) \text{ whence}$$
$$x = log\,(u_1 + \sqrt{u_1^2 + 1}) \tag{12.1}$$

Similarly for u_2 $$x = log\,(u_2 + \sqrt{u_2^2 - 1}) \tag{12.2}$$

Then for u_3 we have again from (10.3) by division

$$e^{2\,x} = \frac{1 + tanh\,x}{1 - tanh\,x} = \frac{1 + u_3}{1 - u_3}$$

or $$x = \frac{1}{2}\,log\left(\frac{1 + u_3}{1 - u_3}\right) \tag{12.3}$$

These give, therefore, the values of x in terms of the $sinh\,x$, $cosh\,x$ and $tanh\,x$.

13. Derivatives of Hyperbolic Functions. If we express the values of *sinh x*, *cosh x*, *tanh x*, etc. in terms of e^x and e^{-x} and then apply the usual rules for differentiation, we shall find immediately

$$\frac{(d\, sinh\, x)}{d\,x} = cosh\, x$$

$$\frac{d\,(cosh\, x)}{d\,x} = sinh\, x$$

$$\frac{d\,(tanh\, x)}{d\,x} = sech^2\, x$$

14. Illustrations of Complex Functions. At this point, some examples of functions in the form of (2.2) will be of interest, both in themselves and as showing the manner of dealing with different types of functions of this character. Later reference will be made to several of these functional forms.

(a) $$w = \frac{1}{z} = \frac{1}{x + i\,y} = \varphi + i\,\psi$$

$$\frac{1}{x + i\,y} = \frac{x - i\,y}{x^2 + y^2} = \varphi + i\psi\,.$$ Hence

$$\varphi = \frac{x}{x^2 + y^2}$$

$$\psi = \frac{-y}{x^2 + y^2}$$ and

$$w = \frac{x}{x^2 + y^2} - \frac{i\,y}{x^2 + y^2}$$ or if we put $x^2 + y^2 = r^2$

$$w = \frac{x}{r^2} - \frac{i\,y}{r^2}$$ or again transforming wholly to polar

coordinates: $$w = \frac{cos\,\theta}{r} - \frac{i\,sin\,\theta}{r} = \frac{1}{r}\,e^{-i\theta} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

(b) $$w = log\,z = log\,(x + i\,y) = log\,re^{i\theta}$$

or $$w = log\,r + i\,\theta = \varphi + i\,\psi$$

whence $$\varphi = log\,r = log\,\sqrt{x^2 + y^2}$$

$$\psi = \theta = tan^{-1}\frac{y}{x}$$

and $$w = log\,z = log\,r + i\,tan^{-1}\frac{y}{x}$$

(c) $$w = sin\,z = sin\,(x + i\,y) = \varphi + i\,\psi$$

Expanding: $$sin\,x\,cos\,i\,y + cos\,x\,sin\,i\,y = \varphi + i\,\psi$$

or $$sin\,x\,cosh\,y + i\,cos\,x\,sinh\,y = \varphi + i\,\psi$$ (see **10**)

whence $$\varphi = sin\,x\,cosh\,y$$

$$\psi = cos\,x\,sinh\,y$$

(d) $$w = cos\,z = cos\,(x + i\,y) = \varphi + i\,\psi$$

$$cos\,x\,cos\,i\,y - sin\,x\,sin\,i\,y = \varphi + i\,\psi$$

whence as before
$$\varphi = \cos x \cosh y$$
$$\psi = -\sin x \sinh y$$

(e)
$$w = \tan z = \tan(x + i\,y) = \varphi + i\,\psi$$
$$\frac{\tan x + \tan i\,y}{1 - \tan x \tan i\,y} = \varphi + i\,\psi,$$
$$\frac{\tan x + i \tanh y}{1 - i \tan x \tanh y} = \varphi + i\,\psi,$$
$$\varphi + i\,\psi = \frac{(\tan x + i \tanh y)(1 + i \tan x \tanh y)}{1 + \tan^2 x \tanh^2 y}$$
$$= \frac{\tan x + i \tan^2 x \tanh y + i \tanh y - \tan x \tanh^2 y}{1 + \tan^2 x \tanh^2 y}$$
$$\varphi = \frac{\tan x\,(1 - \tanh^2 y)}{1 + \tan^2 x \tanh^2 y}$$
$$\psi = \frac{\tanh y\,(1 + \tan^2 x)}{1 + \tan^2 x \tanh^2 y}$$

(f)
$$w = e^z = e^{(x + i y)} = \varphi + i\,\psi$$
$$e^x e^{i y} = \varphi + i\,\psi$$
$$e^x (\cos y + i \sin y) = \varphi + i\,\psi$$
$$\varphi = e^x \cos y$$
$$\psi = e^x \sin y$$

(g)
$$w = \log \sin(x + i\,y) = \varphi + i\,\psi$$
$$\sin x \cos i\,y + \cos x \sin i\,y = e^{\varphi} e^{i\psi}$$
$$\sin x \cosh y + i \cos x \sinh y = e^{\varphi} (\cos \psi + i \sin \psi)$$
$$e^{\varphi} \cos \psi = \sin x \cosh y$$
$$e^{\varphi} \sin \psi = \cos x \sinh y$$
$$\psi = \tan^{-1} (\cot x \tanh y)$$
$$\varphi = \frac{1}{2} \log \left[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \right]$$

CHAPTER II

INTEGRATION OF PARTIAL DERIVATIVE EXPRESSIONS

1. The integration of partial differential expressions is best approached through a study of the manner in which such expressions are formed.

If we take a function $\quad z = a\,x^3\,y^2$

we have $\quad dz = 3\,a\,x^2\,y^2\,d\,x + 2\,a\,x^3\,y\,d\,y$

Then with the accepted notation we put $\partial z/\partial x$ for $3\,a\,x^2\,y^2$ and $\partial z/\partial y$ for $2\,a\,x^3\,y$ and write in symbolic form:

$$dz = \frac{\partial z}{\partial x} d\,x + \frac{\partial z}{\partial y} d\,y$$

If now we are given $$\frac{\partial z}{\partial x} = 3ax^2y^2$$
it is clear that to find the expression giving rise to this partial differential coefficient, we must simply reverse the process which gave it birth. We must integrate with reference to x and x alone. This will obviously give
$$z = a\,x^3\,y^2$$
If likewise we should be given
$$\frac{\partial z}{\partial y} = 2ax^3y$$
we shall naturally find the primitive of this expression in the same way by integrating solely with reference to y. Carrying this out, we have $a\,x^3\,y^2$ as before.

Thus in the case of a term or terms in z, such as that above, involving both x and y, the integration of either partial derivative relative to the variable appearing in the denominator of its symbolic form, will give the complete expression for z, insofar as such terms are concerned.

Suppose, however, that we have
$$z = x^2 + y^3$$
Then taking partial derivatives we have:
$$\frac{\partial z}{\partial x} = 2x$$
$$\frac{\partial z}{\partial y} = 3y^2$$

Here it is obvious that by integrating $\partial z/\partial x$ we can only get the term x^2 while by integrating $\partial z/\partial y$ we get only y^3.

In such case, therefore, we cannot get the complete function from either partial derivative above, but must have both and use both.

Otherwise, we may say that a partial derivative may be a complete specification of its primitive and from which the primitive may be derived, but it is not necessarily so; and in order to be sure that we have the complete function, z, we must have partial derivatives from both variables and we must integrate both, each with respect to its own variable, and then *omitting duplicates*, we can write the complete function.

Thus for illustration: $$\frac{\partial z}{\partial x} = 4ax^3 + 2bxy^2 + 2cxy$$
$$\frac{\partial z}{\partial y} = 2bx^2y + cx^2 + 3dy^2 + e$$

Then integrating first with reference to x we have:

$z_1 = ax^4 + bx^2y^2 + cx^2y$ and then with reference to y,
$$z_2 = bx^2y^2 + cx^2y + dy^3 + ey$$

Then omitting duplicates we have
$$z = ax^4 + bx^2y^2 + cx^2y + dy^3 + e\,y$$

Or again
$$\frac{\partial z}{\partial x} = \frac{1}{x} + 2x$$
$$\frac{\partial z}{\partial y} = \frac{1}{y} + 2y$$
Then
$$z_1 = \log x + x^2$$
$$z_2 = \log y + y^2$$
$$z = z_1 + z_2 = \log x y + x^2 + y^2$$

CHAPTER III

FOURIER SERIES

1. Fourier Series. In many of the problems of mechanics, especially those involving periodic phenomena, it is found possible to represent a desired quantity, to a continually increasing degree of approximation, as the sum of a series of terms involving functions of successive multiples of an angular quantity defined by the conditions of the problem.

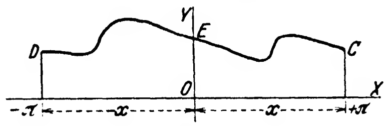

Fig. 5.

Thus the complete development takes the form

$$\left.\begin{aligned} f(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + \\ + B_1 \sin\theta + B_2 \sin 2\theta + B_3 \sin 3\theta + \end{aligned}\right| \tag{1.1}$$

The right hand side of this equation will evidently have equal values for $\theta = -\pi$ and $+\pi$.

It is, in fact, suited to the expression of any such function as in Fig. 5, where the values for $-\pi$ and $+\pi$ are equal. If this should be a periodic function repeating the pattern $C\,E\,D$ indefinitely, then the series as determined for this pattern will hold indefinitely. If, however, the function changes to a different pattern outside the range $-\pi \ldots\ldots +\pi$, then the series as developed will hold only between these limits.

Assuming, however, that the series is applicable in any particular case, our present interest lies in the methods for the determination of the coefficients A and B in the expansion.

This will be found to depend on the integration between limits of $-\pi$ and $+\pi$, of a series of expressions as follows:

$$\cos n\theta\, d\theta \tag{a}$$

$$\sin n\theta\, d\theta \tag{b}$$

$$\cos^2 n\theta\, d\theta = \frac{1}{2}(1 + \cos 2n\theta)\, d\theta \tag{c}$$

$$\sin^2 n\theta\, d\theta = \frac{1}{2}(1 - \cos 2n\theta)\, d\theta \tag{d}$$

$$\sin m\theta \cos m\theta\, d\theta = \frac{1}{2}\sin 2m\theta\, d\theta \tag{e}$$

$$sin\ m\theta\ sin\ n\theta\ d\theta = \frac{1}{2}\left[cos\ (m-n)\ \theta - cos\ (m+n)\ \theta\right] d\theta \qquad \text{(f)}$$

$$cos\ m\ \theta\ cos\ n\ \theta\ d\ \theta = \frac{1}{2}\left[cos\ (m-n)\ \theta + cos\ (m+n)\ \theta\right] d\theta \qquad \text{(g)}$$

$$sin\ m\theta\ cos\ n\ \theta\ d\ \theta = \frac{1}{2}\left[sin\ (m+n)\ \theta + sin\ (m-n)\ \theta\right] d\theta \qquad \text{(h)}$$

The equality in (c) and (d) follows from the expression of $cos\ \theta/2$ and $sin\ \theta/2$ in terms of $cos\ \theta$; that in (*e*) from the expression of sin $2\ \theta$ in terms of sin θ and $cos\ \theta$. The equalities in (f), (g), and (h) are readily established by expanding the right hand members in the form:

$$cos\ (m-n)\ \theta = cos\ (m\theta - n\theta) = cos\ m\theta\ cos\ n\ \theta + sin\ m\theta\ sin\ n\ \theta$$

and similarly for the other expressions.

Expressions (a) and (b) integrate respectively as $(sin\ n\ \theta)/n$ and $-(cos\ n\ \theta)/n$ and these will vanish between the limits of $-\pi$ and $+\pi$.

Expressions (c) and (d) will each evaluate as π.

Expressions (e), (f), (g), and (h) will all integrate and evaluate as *zero* for the same reason as in (a) and (b).

Having these results in mind, we may then proceed as follows:

To determine A_0, integrate (1.1) between the limits of $-\pi$ and $+\pi$. All terms will vanish except the first and we shall have

$$\int_{-\pi}^{+\pi} f\ (\theta)\ d\ \theta = 2\pi A_0 \text{ or } A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f\ (\theta)\ d\ \theta \qquad (1.2)$$

To determine A_n, any coefficient in the A series, multiply (1.1) by $cos\ n\ \theta$ and integrate as before between limits of $-\pi$ and $+\pi$.

Again all terms will vanish except that in $cos^2\ n\ \theta$ and this will give

$$\int_{-\pi}^{+\pi} f\ (\theta)\ cos\ n\ \theta\ d\ \theta = \pi A_n \text{ or } A_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f\ (\theta)\ cos\ n\ \theta\ d\ \theta \qquad (1.3)$$

Similarly, multiplying (1.1) by $sin\ n\ \theta$ we shall have

$$\int_{-\pi}^{+\pi} f\ (\theta)\ sin\ n\ \theta\ d\ \theta = \pi B_n \text{ or } B_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f\ (\theta)\ sin\ n\ \theta\ d\ \theta \qquad (1.4)$$

If, therefore, the function is known between limits of $-\pi$ and $+\pi$ for θ (corresponding to $-x$ and $+x$ as in Fig. 5) the various coefficients may be found by any suitable process of integration.

2. Fourier Series Continued. We have seen that a Fourier expansion as in (1.1) is applicable only when the values at C and D Fig. 5 are the same. This is obvious from the form of the equation.

Let us now consider two special cases as represented in Fig. 6. First take the function represented by $C\ E\ D$, symmetrical about y. Evidently in such case, $f\ (\theta) = f\ (-\theta)$. Likewise we have $sin\ n\theta = -sin\ (-n\theta)$.

Hence in the formula for a B coefficient, the elements in the integration will form pairs equal in value and opposite in sign and thus cancelling out in the summation. Hence each B coefficient will vanish. Again since $\cos n\theta = \cos(-n\theta)$, the elements in the integration for an A coefficient will form pairs equal in value and of the same sign and thus combining in the summation. The result for the entire integration from $-\pi$ to $+\pi$ will be, therefore twice that from 0 to π. Hence for the A coefficients we shall have

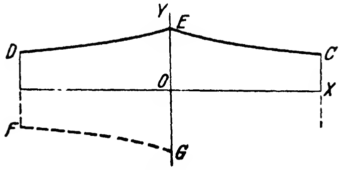

Fig. 6.

$$A_n = \frac{2}{\pi}\int_0^{\pi} f\,\theta\,(\cos)\,n\,\theta\,d\theta \qquad (2.1)$$

Due to the symmetry about Y we shall also have for the A_0 term:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta \qquad (2.2)$$

In such case, therefore, and between the limits for θ of 0 and π, we may write

$$f(\theta) = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta + \Sigma\left[\frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\,\theta\,d\theta\right]\cos n\,\theta \qquad (2.3)$$

where n is given successive values as may be required by the nature of the approximation desired.

Suppose again a discontinuous form of the function as represented by $CEGF$, the branch GF being symmetrical about X with ED. Then with reference to the relation on the two sides of the y axis, we shall have $f(\theta) = -f(-\theta)$. In such case, in the formula for an A coefficient, the elements in the integration will form pairs equal in value and with opposite signs and hence the A coefficients will vanish. On the other hand the elements for the B coefficients will form pairs equal in value and with the same sign, and hence they will combine in the summation.

In like manner as for (2.1), this will give, between the limits 0 and $+\pi$,

$$f(\theta) = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta + \Sigma\left[\frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\,\theta\,d\theta\right]\sin n\,\theta \qquad (2.4)$$

We have thus, three forms of expansion as in (1.1), (2.3), (2.4).

In (1.1) the coefficients are to be determined as in (1.3) and (1.4) and its application is limited to cases where the two end ordinates are equal.

In (2.3) and (2.4) the end ordinates need not be equal and these forms are therefore applicable to any form or pattern of function between these limits.

We have next to show how we may adjust any limits of a function $y = f(x)$ in such manner as to suit these limits for θ.

Suppose, as in Fig. 7, we have given any function

$$y = f(x)$$

between limits x_1 and x_2. We wish x_1 to correspond to $\theta = 0$ and x_2 to $\theta = \pi$. We have then simply,

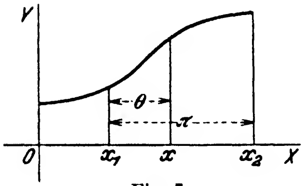

Fig. 7.

$$\theta = \pi \frac{(x - x_1)}{(x_2 - x_1)} \tag{2.5}$$

We may then take successive values of x and determine θ or *vice versa* as we choose. This makes possible, therefore, the application of a Fourier expansion over any range and for any function the values of which are known over the range in question.

The application of these methods of expansion may be illustrated by a simple example.

Given the equation $\quad y = \sqrt{x+5} - 1$

This is, of course, an arc of a parabola, see the full line, Fig. 8. Let us now attempt to represent this line between the values for x of 0 and 10 by a Fourier series, as in (2.3).

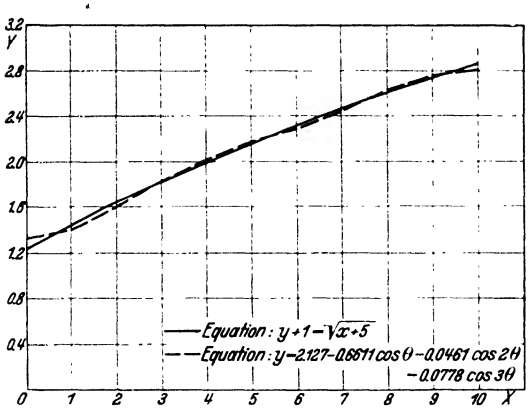

Fig. 8. Comparison of actual values with development in Fourier series.

We must first establish the relation between x and θ in the formula. This we do by (2.5) giving $\theta = \pi\, x/10$.

We next find, either by planimeter or by numerical integration, the mean ordinate of this curve. This will give A_0. Numerical integration by the trapezoidal rule gives $A_0 = 2.127$.

We next multiply each value of $y = f(\theta)$ by the value of $\cos \theta$. Thus for $x = 3$ we have $y = 1.828$ and $\theta = 54^\circ$. This gives $y \cos \theta = 1.074$ and similarly for the others.

Summing these as in (2.1) we find $A_1 = -.6611$. We then repeat for 2θ. In this case $2\theta = 108^\circ$ and for $x = 3$, $y \cos 2\theta = -.5649$. Summing again as in (2.1) we find $A_2 = -.0461$ and similarly $A_3 = -.0778$.

Quantity	*Dimensions*
Velocity: length $\div$ time	$L\,T^{-1}$
Angular Velocity	T^{-1}
Revolutions in unit time: number $\div$ time	T^{-1}
Acceleration: velocity $\div$ time	$L\,T^{-2}$
Momentum: mass $\times$ velocity	$M\,L\,T^{-1}$
Force: mass $\times$ acceleration	$M\,L\,T^{-2}$
Moment of momentum: momentum $\times$ length	$M\,L^2\,T^{-1}$
Moment of force (Torque): force $\times$ length	$M\,L^2\,T^{-2}$
Energy, Work: force $\times$ length	$M\,L^2\,T^{-2}$
Power: Work per unit of time	$M\,L^2\,T^{-3}$
Pressure: force $\div$ area	$M\,L^{-1}\,T^{-2}$
Pressure gradient along a length: pressure $\div$ length	$M\,L^{-2}\,T^{-2}$
Viscosity: force per unit area $\div$ rate of shear	$M\,L^{-1}\,T^{-1}$
Kinematic Viscosity: viscosity $\div$ density	$L^2\,T^{-1}$
Vorticity: velocity $\times$ length	$L^2\,T^{-1}$

Of these various items, viscosity may require a word of further explanation. To develop the basic concept in the measure of viscosity, picture an indefinite plane parallel to which a plane of area a is moving with a velocity v and with a separation of h between the planes, the space between being filled with the fluid of which the viscosity is to be measured. The force due to viscosity varies directly as the area and as the relative velocity of sliding (rate of shear) between adjacent layers, and the latter is assumed to be distributed uniformly through the thickness of the film of fluid. The total relative velocity between the two sides of the film is v and the thickness is h. Hence the rate of shear is v/h. Since then the total force varies as the area and as the rate of shear, the unit force will be that for a unit area moving with a unit rate of shear. This will be given by dividing the total force F by the area a and by the rate of shear v/h. Hence the unit of viscous force involves the operations expressed by the formula

$$\frac{F}{a\,v/h} = \frac{F\,h}{a\,v}$$

and putting in the dimensions of a force, a length, an area, and a velocity, we find the dimensions as in the table.

The theory of dimensions has several important applications in connection with the various problems of mechanics.

It serves to determine the relation between the magnitude of the various units of velocity, acceleration, force, pressure, work, etc. as dependent on different values of the fundamental units of length, mass and time, as for example, between the English and the metric systems of measure.

It may be used as a means of testing the consistency of equations involving physical quantities. Thus in the case of an equation in algebraic form expressing the relation between physical quantities and perhaps involving several terms, it is clear that if the equation is rational

and consistent, all of these terms must imply the same physical quantity. Thus we cannot have momentum and energy as terms in the same equation. We cannot add together velocity and density or force and mass. It results that all terms in such an equation must have the same dimensions and the application of this test is often useful for the detection of errors in transformation or manipulation[1].

2. Kinematic Similitude. The most important application of the theory of dimensions to the problems of aeronautics is found in the development of the theory of kinematic similitude. Of this we shall give here only a brief outline.

We must first note the characteristics of a dimensionless quantity. Algebraically, when the expression of the dimensions reduces to unity, the quantity is dimensionless. Thus

Length ÷ Length $= L \div L$
Volume ÷ Statical Moment of Area $= L^3 \div L^3$
Moment of force ÷ Energy $= M L^2 T^{-2} \div M L^2 T^{-2}$

Thus a ratio is always dimensionless and generally the quotient of two quantities each of which has the same dimensions.

Again take the expression $L g v^{-2}$ in which L is a length, g is acceleration and v is velocity. If the dimensions are put in, they will be found to reduce to zero, showing the quantity to be dimensionless.

Again if any quantity is dimensionless, it is clear that its reciprocal will be so likewise, and also the quantity or its reciprocal affected by any exponent m.

Thus the expressions:

$$\frac{Lg}{v^2}, \qquad \frac{v^2}{Lg}, \qquad \left(\frac{Lg}{v^2}\right)^m, \qquad \left(\frac{v^2}{Lg}\right)^m$$

are all dimensionless.

3. The Π Theorem. The basic theorem upon which rests the application of the principles of kinematic similitude to problems in mechanics is often known as the *Π theorem*. It is in effect a special form of statement of the principle of dimensional homogeneity in all the terms of a physical equation. A statement and form of proof of this important theorem follow.

We assume a natural phenomenon as depending on a number of parameters or conditions. Thus with a fluid moving through a pipe or tubular channel, the loss of head will depend upon:

length of pipe	density of fluid
diameter of pipe	viscosity of fluid
	velocity of flow

[1] In this statement, no consideration is given to equations which may be formed by adding together two separate consistent physical relations such as $v = g t$ and $s = (1/2) g t^2$, as in the laws of falling bodies. In such a case we shall have $v + s = g t (1 + t/2)$, a true but not a normal homogeneous equation. BRIDGMAN, P. W., "Dimensional Analysis". p. 42.

These are all quantities which admit of measure in terms of known units. The loss of head will also depend on the character of the surface of the pipe — its degree of roughness or smoothness — but for this we have no direct measure and we must, therefore, assume simply a standard or corresponding degree of roughness over the field to which the principles of kinematic similitude are to be applied.

In any such case involving a relation between physical quantities, there will be a certain minimum number which will be necessary and sufficient for the definition of all the others. These must, of course, be independent. The selection of the particular group (composed of this minimum number) to be taken for this purpose, is usually a matter of choice.

In mechanics, in the broad sense, the number is three.

For the present assume such number and suppose, for example, the total number five.

Let these be denoted by

$$Q_1,\ Q_2,\ x,\ y,\ z$$

the three latter representing the fundamentals.

Suppose now the relation between these five variables to be expressible in the form of a general algebraic equation. This expression may consist of any number of terms, but from the law of dimensional homogeneity, all terms must have the same dimensions.

The equation may be expressed in the form

$$f\,(Q_1,\ Q_2,\ x,\ y,\ z) = 0$$

Suppose it all written out. Then divide through by any one term, say the first. The result will be a general form of the physical equation with unity for the first term followed by a series of terms in Q_1, Q_2, x, y, z, all of dimension zero.

A typical term in such an equation will be

$$Q_1^a\, Q_2^b\, x^\alpha\, y^\beta\, z^\gamma \tag{3.1}$$

Consider the three expressions

$$Q_1^{a_1}\, x^{\alpha_1}\, y^{\beta_1}\, z^{\gamma_1} \tag{3.2}$$
$$Q_2^{b_1}\, x^{\alpha_2}\, y^{\beta_2}\, z^{\gamma_2} \tag{3.3}$$
$$Q_1^a\, Q_2^b\, x^\alpha\, y^\beta\, z^\gamma \tag{3.4}$$

Of these, (3.2) and (3.3) contain four variables each, *viz.*, the three fundamental reference variables x, y, z, with, in each case, one of the remaining variables Q; while (3.4) is the typical term as in (3.1), and must be of zero dimension. Let us assume, for the moment, that (3.2) and (3.3) are also of zero dimension.

Again let the exponents of L, M, T, in the dimensional expressions for Q_1, Q_2, x, y, z, be as follows:

	L	M	T
Q_1	m_1	m_2	m_3
Q_2	n_1	n_2	n_3
x	p_1	p_2	p_3
y	q_1	q_2	q_3
z	r_1	r_2	r_3

That is, the dimensions of Q_1 are $L^{m_1} M^{m_2} T^{m_3}$ and similarly for the others. Then in order that the dimensions of an expression such as those above shall be zero, it is necessary that the resultant exponents of L, M, T in the final dimensional form shall each independently be zero. This condition applied to (3.2) (3.3) (3.4) will give rise to three sets of simultaneous equations as follows:

$$\left.\begin{aligned} p_1 \alpha_1 + q_1 \beta_1 + r_1 \gamma_1 &= -m_1 a_1 \\ p_2 \alpha_1 + q_2 \beta_1 + r_2 \gamma_1 &= -m_2 a_1 \\ p_3 \alpha_1 + q_3 \beta_1 + r_3 \gamma_1 &= -m_3 a_1 \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} p_1 \alpha_2 + q_1 \beta_2 + r_1 \gamma_2 &= -n_1 b_1 \\ p_2 \alpha_2 + q_2 \beta_2 + r_2 \gamma_2 &= -n_2 b_1 \\ p_3 \alpha_2 + q_3 \beta_2 + r_3 \gamma_2 &= -n_3 b_1 \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} p_1 \alpha + q_1 \beta + r_1 \gamma &= -(m_1 a + n_1 b) \\ p_2 \alpha + q_2 \beta + r_2 \gamma &= -(m_2 a + n_2 b) \\ p_3 \alpha + q_3 \beta + r_3 \gamma &= -(m_3 a + n_3 b) \end{aligned}\right\} \tag{3.7}$$

In each of these sets of equations, the first is formed for the fundamental dimension L, the second for M and the third for T. We have so far simply assumed (3.2) and (3.3) to be of zero dimension. We must now show that it is always possible to write at will such expressions with dimensions zero.

In expression (3.2) for example, suppose a_1 taken arbitrarily at any assumed or convenient value. Then in (3.5), the quantities on the right will be known and we shall have a set of three simultaneous equations with α_1, β_1 and γ_1 as the unknowns. These equations may be solved in the usual manner and will give values of α_1, β_1 and γ_1 which will fulfill the conditions of (3.5) and hence will give zero dimension for expression (3.2). The same is obviously true for expression (3.3). We may therefore conclude that it is always possible to write such expressions with four variables and with exponents such that the dimensions of the expression will be zero. We then assume definitely that the various exponents in (3.2) and (3.3) are determined in accordance with (3.5) and (3.6), and hence that the expressions (3.2) and (3.3) are of zero dimension. Expression (3.4) is already of zero dimension, the conditions for which are expressed in (3.7)

Consider next the two factors and their product:

$$(Q_1^{a_1}\, x^{\alpha_1}\, y^{\beta_1}\, z^{\gamma_1})^{a/a_1}\,(Q_2^{b_1}\, x^{\alpha_2}\, y^{\beta_2}\, z^{\gamma_2})^{b/b_1} = Q_1^a\, Q_2^b\, x^u\, y^v\, z^w \qquad (3.8)$$

where

$$u = \left(\frac{a\,\alpha_1}{a_1} + \frac{b\,\alpha_2}{b_1}\right)$$

$$v = \left(\frac{a\,\beta_1}{a_1} + \frac{b\,\beta_2}{b_1}\right)$$

$$w = \left(\frac{a\,\gamma_1}{a_1} + \frac{b\,\gamma_2}{b_1}\right)$$

It will be noted that the exponents a/a_1 and b/b_1 are so chosen as to give, in the product, the exponents a and b for Q_1 and Q_2, the same as in expression (3.4). We have now to show that the exponents u, v, w, with values as above, will also be equal to the exponents α, β, γ of expression (3.4). That is, we wish to show that the product of the two factors (3.2) and (3.3), the first affected by the exponent a/a_1 and the second by the exponent b/b_1, will reproduce the entire expression (3.4) or (3.1), as representing any given term in the general equation of zero dimension.

To this end we shall state, as below, certain properties of groups of equations such as (3.5), (3.6), (3.7). These properties are either self evident, or the interested reader will readily supply a proof. It may be noted that with an elementary acquaintance with the properties of determinants and their relations to the solution of such equations, the various statements made will be, in effect, self evident.

(1) Considering the quantities α, β, and γ as the variables, the coefficients, represented by the subscript letters p, q, and r, are the same in all three sets.

(2) The relative values of any set of the variables, such as α_1, α_2, α, will therefore depend only on the differences in the known quantities on the right hand side of the equation.

(3) If the known terms on the right hand side of such a set of equations be multiplied by any factor, at will, the value of all the unknowns will be multiplied by the same factor.

(4) If therefore the knowns on the right of set (3.5) are made $-m_1 a$, $-m_2 a$, $-m_3 a$, the values of the unknowns will be increased in the ratio a/a_1.

(5) Similarly if the knowns on the right of set (3.6) are made $-n_1 b$, $-n_2 b$, $-n_3 b$, the values of the unknowns will be increased in the ratio b/b_1.

(6) In a set of equations such as (3.7), with the knowns made up of two parts, the values of the unknowns will be the sum of the two values which would result from two sets of equations, one with the terms in a for the knowns and the other with the terms in b.

(7) It follows that the value of any root, such as α, of (3.7) will be the sum of the two corresponding roots of two sets such as (3.5) and (3.6), but with a instead of a_1 and b instead of b_1.

(8) Hence for the values of the unknowns in (3.7) we shall have as follows:

$$\alpha = \frac{a\alpha_1}{a_1} + \frac{b\alpha_2}{b_1}$$

$$\beta = \frac{a\beta_1}{a_1} + \frac{b\beta_2}{b_1}$$

$$\gamma = \frac{a\gamma_1}{a_1} + \frac{b\gamma_2}{b_1}$$

Hence the term resulting from the product of the two factors as in (3.8), will, in fact, reproduce the term (3.1) of the general equation of zero dimension.

Factors of the form of (3.2) and (3.3), comprising the three fundamental variables x, y, z, with one of the other variables Q, and of zero dimension, may be represented by the symbol Π and thus we may denote the two factors in (3.8) by $(\Pi_1^{a/a_1}, \Pi_2^{b/b_1})$ or more generally by $(\Pi_1^{\lambda}\, \Pi_2^{\mu})$ where λ and μ are chosen according to the particular term of the general equation which is to be represented. It thus appears that the general equation, consisting of a series of terms all of zero dimension, may be also represented as a series of terms all of the form $(\Pi_1^{\lambda}\, \Pi_2^{\mu})$, with the exponents λ and μ varying according to the exponents of Q_1 and Q_2 in the individual term. But any such series of terms, and hence the general equation, may be represented mathematically in the form $f(\Pi_1\, \Pi_2) = 0$.

This is the so-called Π theorem which plays so important a part in many problems involving dimensions and kinematic similitude. It is the equivalent in mathematical language of the statement that any algebraic form expressing a relation between and among a related series of physical quantities, may be adequately and fully represented as a series of terms involving Π functions as defined above, and in the manner indicated in (3.8). Or otherwise, it is equivalent to the statement that each of the terms in a physical equation consisting of a series of terms all of zero dimension may be broken up into Π factors as in (3.2) and (3.3), affected with suitable exponents as the individual term may require. It will be noted that the general form of the physical equation of zero dimension will contain unity as one of its terms, resulting from dividing through the general equation, not of zero dimension, by some one of its terms. This term unity, however, is, of course, of zero dimension and admits of representation by a term of the form $(\Pi_1^{\lambda}\, \Pi_2^{\mu})$ simply by making λ and μ both zero.

Furthermore, it is clear that in any such mathematical expression as $f(\Pi_1\, \Pi_2) = 0$, we may always assume some form of solution for Π_1

in terms of Π_2 or of Π_2 in terms of Π_1 and therefore we may write the equation with equal generality in the form $\Pi_1 = f_1(\Pi_2)$ or $\Pi_2 = f_2(\Pi_1)$. Or otherwise, any such form of statement as $f(\Pi_1 \Pi_2) = 0$ implies a relation between Π_1 and Π_2 which may be expressed in either form $\Pi_1 = f_1(\Pi_1)$ or $\Pi_2 = f_2(\Pi_1)$.

It will also be noted that the number of Π factors will equal the number of Q variables in the general term and this will be three less than the total number of variables involved in the problem. If therefore the problem involves in general n variables, there will be $(n - 3)$ Π factors and the general Π equation may be written in the form

$$f(\Pi_1, \Pi_2, \Pi_3 \ldots\ldots) = 0 \tag{3.9}$$

And here again any such relation may be expressed with equal generality in the form

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4 \ldots\ldots) \text{ or} \tag{3.10}$$

$$\Pi_2 = f(\Pi_1, \Pi_3, \Pi_4 \ldots\ldots) \text{ etc.} \tag{3.11}$$

Equations (3.9), (3.10), (3.11), may be considered as expressions of the Π theorem in its general form.

4. Non-Dimensional Coefficients. The application of these general principles may be illustrated by a simple example using the flow of water in a conduit as referred to in **3**. Listing again the factors and elements with which we are concerned in a problem of this character we have

Loss of head or pressure in conduit	h
Length of conduit	L
Diameter (conduit assumed circular in section)	D
Density of fluid	ϱ
Viscosity of fluid	μ
Velocity of flow	v

Here are six quantities, but we see immediately (assuming the conduit uniform in section) that h will vary directly with L. We can therefore substitute for these two terms their ratio, the loss per unit length which we denote by G. We have then the five quantities denoted by G, D, ϱ, μ, v. The relation between and among these, in its most general form, may then be represented by

$$F(G, D, \varrho, \mu, v) = 0$$

In accordance with the principles of the Π theorem, we shall have, in such case, two Π functions. The general equation, whatever its form or character, must then admit of expression in the form:

$$f(\Pi_1, \Pi_2) = 0$$

We must now select three of these quantities as our fundamental variables, represented by the x, y, z, of **3**. For this purpose we may

select any three so long as they are independent. Let D, v, ϱ be the group chosen. We shall then have our two Π factors as in (3.2), (3.3) in the form $(G^{a_1}, D^{\alpha_1}, v^{\beta_1}, \varrho^{\gamma_1})$ $(\mu^{b_1}, D^{\alpha_2}, v^{\beta_2}, \varrho^{\gamma_2})$.

As noted in 3 we are at liberty to give to a_1 and b_1 any arbitrary or assumed value, since with the expression once adjusted with exponents giving zero dimension, the expression as a whole may be affected by any exponent at choice or such as to give to G or μ any particular exponent as may be desired. We shall therefore take a_1 and b_1 equal to 1. We then set up a table for the first of these expressions as follows,

	M	L	T
G	1	-2	-2
D		1	
v		1	-1
ϱ	1	-3	

and from this table form dimensional equations as in (3.5):

$$\begin{aligned} 1 \qquad\qquad + \gamma_1 &= 0 \\ -2 + \alpha_1 + \beta_1 - 3\gamma_1 &= 0 \\ -2 \qquad - \beta_1 \qquad &= 0 \end{aligned}$$

Solving these we find $\alpha_1 = 1$, $\beta_1 = -2$, $\gamma_1 = -1$

We thus have
$$\Pi_1 = \frac{DG}{\varrho v^2}$$

In the same manner, for the second expression we find:

$$\Pi_2 = \frac{\mu}{Dv\varrho}$$

Hence the general form of the relation between these various quantities may be put in the form

$$f\left[\frac{DG}{\varrho v^2}, \frac{\mu}{Dv\varrho}\right] = 0 \tag{4.1}$$

or otherwise
$$\frac{DG}{\varrho v^2} = f_1\left(\frac{\mu}{Dv\varrho}\right) \tag{4.2}$$

$$\frac{\mu}{Dv\varrho} = f_2\left(\frac{GD}{\varrho v^2}\right) \tag{4.3}$$

Taking the first of these we have

$$G = f_1\left(\frac{\mu}{Dv\varrho}\right)\frac{\varrho v^2}{D} \tag{4.4}$$

Since moreover we may affect either of these Π expressions with any exponent, we may take the reciprocal and thus write

$$G = f_3\left(\frac{Dv\varrho}{\mu}\right)\frac{\varrho v^2}{D} \tag{4.5}$$

It may be of interest to note results in the case of another selection of variables as basic. Thus assume the group, G μ, v. Then following through in the same way as before we should find as follows:

$$f\left[\frac{D^2 G}{\mu v}, \frac{\varrho^2 v^3}{G\mu}\right] = 0$$

or

$$\frac{D^2 G}{\mu v} = f_1\left(\frac{\varrho^2 v^3}{G\mu}\right)$$

$$\frac{\varrho^2 v^3}{G\mu} = f_2\left(\frac{D^2 G}{\mu v}\right)$$

or

$$D^2 = \frac{\mu v}{G} f_1\left(\frac{\varrho^2 v^3}{G\mu}\right)$$

and

$$\varrho^2 = \frac{G\mu}{v^3} f_2\left(\frac{D^2 G}{\mu v}\right)$$

Returning now to (4.2) we remember that since $\mu/Dv\varrho$ is of zero dimension, so also will be $Dv\varrho/\mu$ and likewise any function of either of these. Any such function may be represented by a single letter and we may therefore write

$$G = C\,\frac{\varrho v^2}{D} \tag{4.6}$$

or

$$C = G \div \frac{\varrho v^2}{D} \tag{4.7}$$

Any quantity such as C representing the values of some function of a Π or non-dimensional expression of the variables, or again as in (4.6) or (4.7) standing as a factor in an equation in which it must be of zero dimension in order to render the equation homogeneous as to dimensions, is known as a "*non-dimensional coefficient*".

Such coefficients play a large part in the application of dimensional analysis to the problems of mechanics and engineering and in fact represent the chief agency for the application of these principles to practical problems in these domains.

The particular Π expression $(Dv\varrho/\mu)$, as developed above, is found to play a role of primary importance in connection with all problems involving the relative motion of fluids and solids. It is commonly called the "*Reynolds number*" from Osborne Reynolds who first drew special attention to its significance in problems of this character. It is sometimes written as above, but more frequently in the form

$$\text{Reynolds Number} = \frac{Dv}{\nu}$$

where ν, called the "*kinematic viscosity*" is put for the ratio μ/ϱ.

As a further example, suppose that we have a family of airfoils, all of the same geometrical form and character but differing in absolute size. Let these be tested for lift in a wind tunnel, all at the same angle of attack. Then assuming that the tunnel is of such size that the airflow about the foil is in no case sensibly affected by the boundary walls of

the tunnel, or otherwise neglecting such influence, we may say that in this entire set of tests, the geometry will be the same throughout, including the form of the foil and the form and character of the fluid flow. Broadly speaking, then, the geometry will be similar throughout and will involve only one variable representing size.

The remaining elements of the problem are clearly as follows:

Lift (force), velocity, density of air, viscosity of air. We may therefore represent the variables as follows:

Force	F	Velocity	v
Dimension	L	Viscosity	μ
		Density	ϱ

As in the previous case we have five variables and hence there will be two Π functions. We shall take L, v, ϱ as the fundamental variables. This will give the two Π expressions in the form:

$$F\,L^{\alpha_1}\,v^{\beta_1}\,\varrho^{\gamma_1} \quad \text{and} \quad \mu\,L^{\alpha_2}\,v^{\beta_2}\,\varrho^{\gamma_2}$$

where, as before, we take the exponents of F and μ as unity. Then by the same procedure as before we find for the general expression in terms of Π functions:

$$f\left[\frac{F}{L^2\,v^2\,\varrho},\quad \frac{L\,v\,\varrho}{\mu}\right] = 0$$

or

$$F = f_1\left(\frac{L\,v\,\varrho}{\mu}\right)\varrho\,L^2\,v^2$$

Here, as before, we might substitute for the first expression a single letter C_L for example, and write the relation in the form

$$F = C_L\,\varrho\,L^2\,v^2$$

Or, again, since L^2 is proportional to area, which we may denote by S, we may write

$$F = C_L\varrho\,S\,v^2 \tag{4.8}$$

Here again, C_L represents a non-dimensional coefficient (really the values of the functions of the Reynolds number) and thus serves to express the value of the force F in terms of the three variables ϱ, S, and v.

One item of special importance in connection with non-dimensional coefficients such as a function of the Reynolds number, is that the value of the function, that is, the value of the coefficient, depends in no wise on the individual values of L, v, ϱ, μ entering into its composition, but only on their group value as given by the expression $Lv\varrho/\mu$. Hence no matter how widely the individual values may vary in a series of cases, if the group value is the same, so also will be the value of the coefficient. Thus for example, supposing ϱ/μ to remain the same, we might have a large value of L and small value of v in one case and a small value of L and large of v in the other, both giving the same product Lv and hence the same Reynolds number and hence the

same value of the coefficient. Or otherwise, any value of the coefficient corresponding to a given value of the Reynolds number will apply to all cases which have the same number, regardless of the individual values of L, v, ϱ, and μ. Still otherwise, the application of these coefficients may be stated as follows. Given a particular case with specified values of L, v, ϱ, and μ. This will give a definite value of the number $L v \varrho/\mu$. If then from measurements perhaps with other substances, other dimensions and other velocities, but with the same value of $L v \varrho/\mu$, we have at hand the values of the coefficient for the latter case, the same coefficient may be employed for the case in hand.

While coefficients arising out of the Reynolds number, $L v \varrho/\mu$, have been used here by way of illustration of the character and significance of such coefficients, the same significance and the same wide character of use attach to all coefficients of this nature, *i. e.*, to all quantities representing the values of functions of non-dimensional or Π groups of variables as developed in **3**.

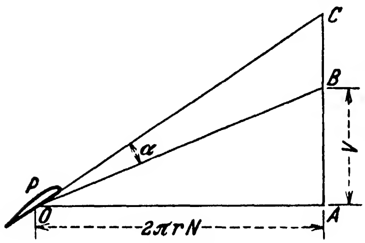

Fig. 9.

Another feature of non-dimensional coefficients of very great importance and utility arises from the fact that being non-dimensional, the value is independent of the particular system of units employed—metric or English. The value of π, the relation between ten feet and two feet, twelve meters and four meters, six per cent—these and other like values are independent of any system of units of measurement, so long as a single consistent system is used in each case. So it is with all non-dimensional quantities. It thus develops that values derived from measurements made with the English system of units and expressed in the form of non-dimensional coefficients, will be identical in numerical value with the results of the same measurements made with the metric system of units, but expressed likewise in the non-dimensional form. In this manner the results of experiments made with one system of units become directly available for use in another system, without the sometimes tedious process of transformation from one system to the other.

One further illustration will be of value. The blade of a propeller is a special form of airfoil moving in a helical path as the plane moves forward in a straight line. Suppose then that we have two propellers of the same geometrical form and proportion but different in size and moving at different speeds. The condition for similarity of air flow in the two cases must first be examined.

In Fig. 9 let P denote any element of the blade looking from the tip toward the hub. Let r denote the radius of this element and N the revolutions in unit time. Then the speed of the element in rotation

will be $2\pi r N$. Let this be represented in direction and magnitude by OA. Let V be the speed of advance, represented likewise by AB. Then the relative speed of the element through the air will be represented in direction and magnitude by OB, and in particular, the angle of attack α will be measured by COB. Now in order that similar elements on propellers of different diameters may have the same angle of attack, it will be necessary and sufficient to have the angle COB, the same in each case. The condition for this may be expressed as $AB/AO =$ *constant* or $V/2\pi ND =$ *constant*. In such an expression, however, the constant 2π may be omitted, leaving the condition in the form

$$\frac{V}{ND} = constant$$

This is the condition for any element of the blade and for the propeller as a whole. The especially significant and important feature of this expression is that it is non-dimensional. The use made of this feature will appear in other Divisions of this work.

Turning now to the element or to the blade as a whole, considered as an airfoil, it is clear that the force reaction between the blade and the air will be of the same character as in the preceding case and that the dimensional equation must be of the same general form.

We shall then be concerned with quantities as follows:

Diameter	$= D$	Thrust (force)	$= T$
Velocity	$= V$	Torque	$= Q$
Density	$= \varrho$	Power	$= P$

Then for the thrust equation we may write

$$T = C_T \varrho D^2 V^2$$

where C_T is a non-dimensional coefficient representing the function of the Reynolds number for this particular case.

We can now derive immediately the form of the corresponding equations for torque and for power. Thus torque is the moment of a force and will have the dimensions of force multiplied by a length. Again power has the dimensions of force multiplied by velocity. If C_Q and C_P denote similarly non-dimensional coefficients for torque and for power, we may write

$$Q = C_Q \varrho D^3 V^2$$

$$P = C_P \varrho D^2 V^3$$

Solving for the various coefficients, we have

$$C_{T1} = \frac{T}{\varrho D^2 V^2}$$

$$C_{Q1} = \frac{Q}{\varrho D^3 V^2}$$

$$C_{P1} = \frac{P}{\varrho D^2 V^3}$$

Now any non-dimensional coefficient may be multiplied or divided by any other non-dimensional coefficient without changing its basic character. We may therefore multiply any or all of the above coefficients by the non-dimensional expression V/ND affected by any exponent at choice. We thus find

$$C_{T2} = \frac{T}{\varrho N^2 D^4} \qquad C_{T3} = \frac{T N^2}{\varrho V^4}$$

$$C_{Q2} = \frac{Q}{\varrho N^2 D^5} \qquad C_{Q3} = \frac{Q N^3}{\varrho V^5}$$

$$C_{P2} = \frac{P}{\varrho N^3 D^5} \qquad C_{P3} = \frac{P N^2}{\varrho V^5}$$

It will be noted that the coefficients with 1 in the subscript are in terms of D and V, those with 2, in terms of N and D and those with 3, in terms of N and V.

Again any of these forms of coefficient may be affected by any exponent, whole or fractional, without in any way changing its basic character. In this way, an indefinite number of such coefficients may be developed, in terms of any pair of the three variables D, V, N, as desired, and of any range of numerical values as may be convenient for the purpose in hand. Thus for example a given coefficient might show values ranging from some moderate numerical value off to ∞. This would be inconvenient for graphical representation. In such case we have only to invert the expression, giving values from some convenient number down to 0. This, to a suitable scale will then give a convenient form of graph.

By making use of these various possibilities, non-dimensional coefficients of the widest range of form and character may be developed according to special choice or convenience.

CHAPTER V

VECTOR ALGEBRA: TWO-DIMENSIONAL VECTORS

Vectors and the algebra of the imaginary $\sqrt{-1}$ find frequent and important application in the theory of aerodynamics.

In the mathematical development of the vector concept, two measurably distinct systems or methods have been followed, according as the vectors are two-dimensional or three-dimensional. Two-dimensional vectors have certain special and important applications in the various problems of aeronautical science. Three-dimensional vectors furnish a most convenient and effective means of approach to the general problems of fluid mechanics and to many of its applications in aeronautic science. The present Division deals with the subject of two-dimensional vectors only. The subject of three-dimensional vectors will be found briefly treated in Division C.

Only the bare outlines of the more fundamental features of the subject can, of course, in either case be presented within the limits of the space here available.

1. Definition of Vector and Scalar. The physical concepts, force, velocity, movement, imply, intrinsically, two specifications each — *magnitude* and *direction*. The term *vector* is used to specify quantities and concepts of this character. They are *vector* quantities.

Now it is seen that a right line furnishes exactly the means for representing geometrically such physical qualities or concepts. Such a line has two specifications—length and direction. The length of the line may represent magnitude to any arbitrary scale and the direction of the line may represent the direction of the line of action of the physical quantity, in space or in a plane, as the case may require.

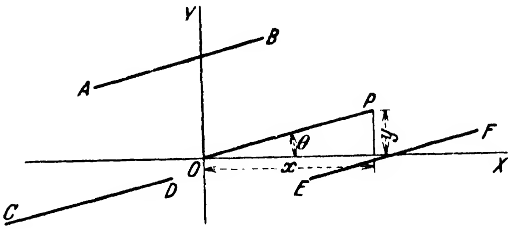

Fig. 10.

A line used in this manner and for this purpose is known as a *vector*. From the geometrical viewpoint, therefore, we may simply consider a vector as a line characterized by the two specifications, *length* and *direction*. It must be especially noted that a vector has this complex make-up. It cannot be represented or defined by either of its specifications taken singly.

A space vector is a length having a specified direction in space. A plane vector is a length having a specified direction in a plane.

In contradistinction to the term *vector*, the term *scalar* is used to imply a number, or geometrically, a length, without reference to direction. The scalar value of a vector is simply its length measured in any convenient unit.

Thus in Fig. 10 if the length of OP is 10 units, then as a scalar, OP is a line 10 units long and that is all; while as a vector, OP is a line 10 units long and inclined at an angle θ with the XX direction.

It may also be noted that the specification *direction* implies something more than *line* of action. It implies also a *sense* or *direction* of action. Thus the vector AB would imply action or movement from A towards B, while the vector BA would imply action or movement from B towards A.

It should also be noted that neither length nor direction, as a specification, will serve to locate the vector in a plane such as XY. A vector as such, therefore, is not located and may lie anywhere in its plane of action. If it is to be located as to position or point of action, this

must be done by locating some specified point on the vector, as for example, the initial end—the end from which the action is assumed to start. Thus in the strict vector sense, in Fig. 10, OP, AB, CD, EF are all equal as vectors provided they have the same length and the same direction.

2. Algebraic Representation of a Vector. While, therefore, the geometrical representation of a vector is thus very simply and completely realized, there is needed something further if we are to realize any particular advantage from the introduction of this concept into the study of physical problems. This must be furnished by an algebra of vectors based upon some mode of algebraic representation. We have need, in fact, for two types of representation, the first of which should be merely a symbol or tag, so to speak, indicating simply that the thing thus designated is a vector; while the second should, in some way, contain within itself the two characteristics of the vector, length and direction, and combined or represented in such algebraic form as to permit of the consistent application of certain algebraic operations.

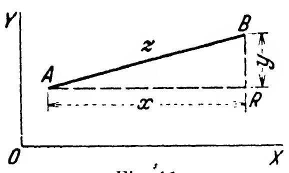

Fig. 11.

For the first type of representation we may use a single letter such as z or ζ or two letters indicating the vector geometrically such as AB Fig. 10. Where a single letter is used it is customary to employ a special type such as z or some other typographical indication. Again the numerical or scalar value of a vector may be indicated by writing the designation between vertical lines thus:

$$|z| = 10 \text{ or } |OP| = 10$$

This is read, scalar value of vector z or of vector OP equals ten.

For the second type of vector representation, two forms are available which we now proceed to develop.

3. Representation by Rectangular Components. The reader is already familiar with the sense in which a force, a velocity, a right line movement, may all be decomposed into components, such as the X and Y components of the line AB, Fig. 11, or again with the sense in which we may recombine such components into the original vector or line.

We may thus say that so far as the action of the physical concept (force, velocity, or right line movement) is concerned, there is a complete equivalence between the force and its components, the velocity and its components or the right line movement and its components. In this same sense, therefore, we may say that there is a complete equivalence between the vector AB and its component vectors AR and RB or otherwise between the vector z and its components x and y.

This equivalence expressed in algebraic form will serve the purpose of complete vector representation. Usage has developed the form:

$$z = x + \sqrt{-1}\, y \text{ or } z = x + i\, y \qquad (3.1)$$

The symbol i is to be here understood simply as an operator or as an indicator of direction. As an operator it implies a rotation through $+90^\circ$. Any letter used as a scalar that is, simply as a measure of length, is understood to be laid off in the direction of the scalar axis which we here take as the axis of X. A vector expressed as $x + i\,y$ is therefore to be understood as directing that a length x be laid off along the direction of X followed by a length y laid off at $+90^\circ$ to the same axis. It should be noted that here, y does not, in itself, imply a length laid off in the direction of y, but simply a length, while the i operator tells us that this length is to be laid off at $+90^\circ$ to the axis of scalars, or here in the direction of $+Y$. In the same way a vector $a + i\,b$ means a length a laid off in any direction chosen as the axis of scalars, followed by a length b laid off at $+90^\circ$ to such axis.

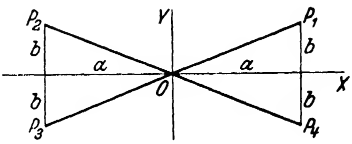

Fig. 12.

The representation of the vector is complete in this form, since we can derive therefrom its two essential characteristics, thus:

$$\text{Length} = \sqrt{x^2 + y^2} \text{ or } \sqrt{a^2 + b^2}$$

Tangent of the angle of inclination to the axis of scalars

$$\tan \theta = y/x \text{ or } b/a$$

Regarding the operator i, the following relations may be noted:

$$i \times i = -1 \text{ or } -i \times i = 1 \text{ and } 1/i = -i$$

Powers of i of the series 0, 4, 8, etc. are $+1$
Powers of i of the series 2, 6, 10, etc. are -1
Powers of i of the series 1, 5, 9, etc. are $+i$
Powers of i of the series 3, 7, 11, etc. are $-i$.

Taking then $z = a + i\,b$ as the general characterization of a vector and referring to Fig. 12, with the usual convention regarding algebraic signs it will be clear that we shall have (taking a and b intrinsically positive) as follows:

$$OP_1 = a + i\,b$$
$$OP_2 = -a + i\,b$$
$$OP_3 = -a - i\,b$$
$$OP_4 = a - i\,b$$

Comparing the first and the third of these or the second and the fourth, it is clear that

$$OP_3 = -OP_1$$
$$OP_4 = -OP_2$$

But OP_3 is OP_1 reversed and similarly for OP_4 and OP_2. It thus appears that changing the sign of a vector reverses it in direction leaving

it otherwise unchanged. This is, of course, entirely consistent with the usual conventions of analytical geometry since $+x$ and $-x$ are equal in amount and reversed in direction. The present development extends this same relation to lines having any direction in the plane. If therefore in Fig. 10, AB means a vector taken in the sense A to B, then $-AB$ means the same vector reversed or taken in the sense B to A. Hence with this understanding regarding the significance of the order of the letter designating a vector, we may write $-AB = BA$ or $-BA = AB$.

Attention may be here drawn to the distinction between the designation of a vector in the form

$$z = x + i\,y \text{ and}$$
$$z = x,\ y$$

where the latter gives merely the x and y components of the scalar length, but without algebraic relation between them. The latter form specifies, indeed, the vector itself as completely as does the former, but that is as far as it goes. It does not make possible the use of the vector in algebraic operations, while the former, as we shall see, does provide for such use and makes possible the effective entry of the vector in this form into algebraic operations of the most varied character.

We now proceed to develop a second mode of representation for a vector.

4. Exponential Representation of a Vector. Referring to I 4, we have, as analytical equivalents

$$\cos\theta + i\sin\theta = e^{i\theta}$$

or

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

But $r\cos\theta$ and $r\sin\theta$ are the direct geometrical equivalents of x and y. Hence we may write the continued equation,

$$z = x + i\,y = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

in the full vector sense, affirming the equality of the three forms in which z is expressed.

It is of interest to note that the equivalence of $(x + i\,y)$ and $r(\cos\theta + i\sin\theta)$ is of the nature of a direct geometrical identity, as above noted, while the equivalence of $(\cos\theta + i\sin\theta)$ to $e^{i\theta}$ is of an analytical character, requiring, as shown in I 4, the expansion of these terms by Maclaurin's theorem in order for the equivalence to become apparent.

We may write, therefore, in the vector sense

$$z = re^{i\theta} \tag{4.1}$$

This is then an alternate form for the representation of a vector, in which r is the scalar value or actual length while θ gives the direction angle.

The representation is thus complete and the two characteristics are related algebraically in such manner, again, as to permit of the entry of the vector in this form into algebraic operations of varied and extended character.

We shall now, through these forms, proceed to develop the more common algebraic operations involving vectors.

5. Addition of Vectors. Taking form (1) in Fig. 13 we have

$$AB = x_1 + i\,y_1$$
$$BC = x_2 + i\,y_2$$

We now apply the usual rules of algebra to the addition of these two vectors. We may evidently do this under the condition that we are able to interpret our results in a manner consistent with the geometry of the diagram. We shall then have:

$$AB + BC = (x_1 + x_2) + i\,(y_1 + y_2)$$

On the right the interpretation is clear. It directs us, starting from A to go a distance $(x_1 + x_2)$ in the X direction and then a distance $(y_1 + y_2)$ in the Y direction. This will obviously take us from A to C.

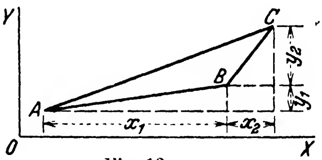

Fig. 13.

Now AB alone implies going from A to B or otherwise it implies A and B as the initial and terminal points of some operation or experience. Likewise BC implies similarly for B and C. Then $AB + BC$ should imply the result of combining these operations in sequence, and this clearly should take us from A to C or otherwise would give A and C as the initial and terminal points of the joint operation.

The interpretation is then consistent since each side of the equation implies the same overall result, or otherwise the same initial and final points. But as a vector we may clearly write also

$$AC = (x_1 + x_2) + i\,(y_1 + y_2)$$

Hence $$AC = AB + BC \tag{5.1}$$

It should be especially noted that this equation can be interpreted only in the vector sense. In the numerical or in the usual algebraic sense, AC does not equal $AB + BC$. But in the vector sense, or in the sense in which AC represents a force, a velocity, a right line movement, while AB and BC represent component forces, velocities or movements, we may write, $$AC = AB + BC$$

It is, of course, seen that this is only an algebraic expression of the parallelogram of forces, velocities, or right line movements. It also simply generalizes the particular case of Fig. 11 which we agreed to write as $$AB = AR + RB$$

or $$z = x + i\,y$$

Law of commutation. In an algebraic sense, we know that:

$$A + B = B + A$$

or again the sum of $A + B + C + \ldots\ldots$ is the same regardless of which one of the various different sequences we may take in associating these letters in a sum. With algebraic quantities, associated in a sum, therefore, the order or sequence is indifferent.

To examine the case for vector quantities we first ask whether, in the vector sense $\qquad AB + BC = BC + AB$

The meaning of this is: Starting from an initial point A and executing first a vector with the specifications of AB and then in sequence a vector with the specifications of BC, thus arriving at C, will the result be the same if instead we should take the vectors in the inverse order?

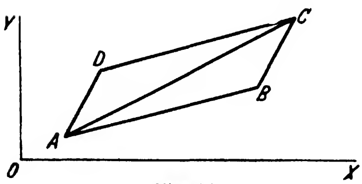

Fig. 14.

In Fig. 14 let AB and BC denote the vectors in question. Complete the parallelogram $ABCD$. Then from the addition theorem:

$$AC = AB + BC$$
$$AC = AD + DC$$

But as vectors, $AD = BC$ and $DC = AB$.
Hence $AB + BC = AD + DC = BC + AB$.

By similar reasoning, this result may be generalized for any number of vectors and taken in any order.

Otherwise we may reach the same result by taking the X and Y components of the various vectors: Thus denoting any vector in general by z we may write:

$$z_1 = x_1 + i\,y_1$$
$$z_2 = x_2 + i\,y_2$$
$$z_3 = x_3 + i\,y_3$$
$$z_4 = x_4 + i\,y_4$$

Then $\boldsymbol{\Sigma} z = \Sigma\, x + i\,\Sigma\, y$.

But in the numerical sense the value of $\Sigma\, x$ will be the same whatever the order in which the individual values are taken, and the same for $\Sigma\, y$. Hence no matter what the order of the vectors, the values of $\Sigma\, x$ and $\Sigma\, y$ will be the same. But these are the X and Y components of the final vector sum $\boldsymbol{\Sigma} z$ and hence the resultant vector $\boldsymbol{\Sigma} z = \Sigma\, x + i\,\Sigma\, y$ will be the same no matter in what order the individual vectors may be taken.

6. Subtraction of Vectors. To the vector equation

$$AC = AB + BC$$

let us apply the usual rules for algebraic transposition:

$$AC - BC = AB \qquad (6.1)$$

This may be rewritten
$$\left.\begin{aligned} AC + (-BC) &= AB \\ AC + CB &= AB \end{aligned}\right\} \tag{6.2}$$

But, as we have seen, CB means BC reversed or referring to Fig. 14 if BC means from B to C then CB means from C to B.

Equation (6.2) is then to be interpreted as stating that the combination of AC and CB is equivalent to AB and in the vector sense this is obviously correct.

Equation (6.1) as a vector subtraction, is therefore correct since it leads to a result consistent with our previous definitions and conventions.

Again suppose instead of (6.1) we write

$$AC - AB = BC$$

This may be rewritten: $AC + BA = BC$

or $BA + AC = BC$

which is obviously correct as a vector equation. The equation $AC - AB = BC$, from which this is derived, is therefore correct and consistent as an expression of vector subtraction.

In general we can always write any vector equation involving combinations in sequence in the form

$$AB + PQ + EF + MN + \ldots\ldots = \textit{some resultant vector } RS$$

or $$AB + PQ + EF + MN + \ldots\ldots + SR = 0.$$

Then in any such equation the elements are subject to the usual algebraic rules of transposition and of association in any manner whatever.

We have thus established the rules for vector addition and subtraction through the use of the first form of vector representation $z = x + iy$. The second or exponential form is not well suited to the discussion of these particular operations. It will, however, find effective application in operations involving multiplication, division, roots and powers, to which we now proceed.

7. Multiplication of a Vector by a Scalar. Let

$$z = x + iy \tag{7.1}$$

Multiply by a number m. Then

$$mz = mx + imy \tag{7.2}$$

It is seen that (7.2) is a vector as in (7.1), only on a scale m times as large. Thus the multiplication of a vector by a scalar simply multiplies the linear dimensions of the vector leaving the direction unchanged.

If m is a multiplier, therefore, the scalar value of the resultant vector will be multiplied by m and the direction angle left the same.

8. Multiplication of a Vector by a Vector. Let us write the vectors:

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$

Then applying the usual algebraic rules for multiplication we shall have:

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i\,(x_1 y_2 + x_2 y_1) \qquad (8.1)$$

This product has the form

$$P = Q + i\,R$$

It has therefore the general form of a vector of which Q is the X component and R the Y component.

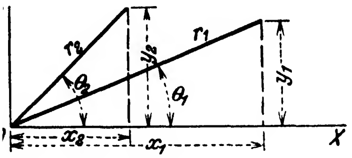

Fig. 15.

We can readily interpret the right hand side of (8.1) by substituting as follows, see Fig. 15.

$$x_1 = r_1 \cos \theta_1 \qquad y_1 = r_1 \sin \theta_1$$
$$x_2 = r_2 \cos \theta_2 \qquad y_2 = r_2 \sin \theta_2$$

Then (8.1) becomes:

$$z_1 z_2 = r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i\, r_1 r_2 (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)$$

or

$$z_1 z_2 = r_1 r_2 \cos (\theta_1 + \theta_2) + i\, r_1 r_2 \sin (\theta_1 + \theta_2) \qquad (8.2)$$

This is obviously a vector of which the X component is $r_1 r_2 \cos (\theta_1 + \theta_2)$ and the Y component, $r_1 r_2 \sin (\theta_1 + \theta_2)$.

But for any vector, such as z_1 in Fig. 16, the X component is the scalar value times the cosine of the direction angle θ_1 and the Y component is the scalar value times the sine of the direction angle θ_1.

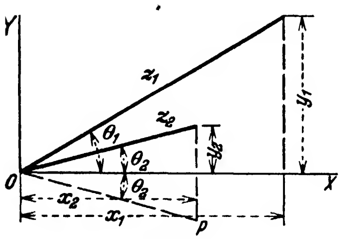

Fig. 16.

Hence in (8.2) the result on the right must be a vector whose scalar value is $r_1 r_2$ and whose direction angle is $(\theta_1 + \theta_2)$.

We have, therefore, the general result that the product of two vectors is a vector of which the scalar value is the product of the scalar value of the two vector factors, and the direction angle is the sum of the direction angles of the two factors.

It is sometimes convenient to consider a vector z_1 as operating on a vector z_2. In such case we may say that the result of the operation is to stretch the vector z_2 from a length r_2 to a length $r_1 r_2$ and to turn it through a positive angle θ_1 thus making the new vector angle $(\theta_1 + \theta_2)$. Similarly we may consider z_2 as operating upon z_1 with the final result the same. In fact, if the preceding detail is followed through, it is readily seen that the order of the factors is indifferent and that

$$z_1 z_2 = z_2 z_1$$

Likewise since the product of two vectors is a vector with specifications as noted, it follows that the product of any number of vectors $z_1 z_2 z_3 \ldots\ldots$ will be a vector with scalar value equal to the scalar product $r_1 r_2 r_3 \ldots\ldots$ and with a direction angle $\Sigma\,\theta$; and furthermore that this

result will be the same independent of the order in which the individual vectors are taken as factors.

9. Division of a Vector by a Vector. Let us indicate the operation in the form
$$\frac{x_1 + i\,y_1}{x_2 + i\,y_2}$$
Now the result of this division must be either a vector or a scalar. The most general assumption is that it will be a vector. Let us write then:
$$\frac{x_1 + i\,y_1}{x_2 + i\,y_2} = a + i\,b$$
Then from the basic concept of multiplication we shall have
$$x_1 + i\,y_1 = (x_2 + i\,y_2)\,(a + i\,b)$$
But this equation describes the vector $x_1 + i\,y_1$ as the result of the multiplication of the two vectors $(x_2 + i\,y_2)$ and $(a + i\,b)$. Hence the scalar value of $(x_1 + i\,y_1)$ is the product of the scalar values of the two vectors $(x_2 + i\,y_2)$ and $(a + i\,b)$. It follows that the scalar value of $(a + i\,b)$ will be the quotient of the scalar value of $(x_1 + i\,y_1)$ divided by the scalar value of $x_2 + i\,y_2$. Again the direction angle of $(x_1 + i\,y_1)$ is the sum of the direction angles of $(x_2 + i\,y_2)$ and $(a + i\,b)$. Hence the direction angle of $(a + i\,b)$ will equal the direction angle of $(x_1 + i\,y_1)$ decreased by that of $(x_2 + i\,y_2)$.

It follows then, similarly as for multiplication, that the quotient of a vector z_1 divided by a vector z_2 will be a vector whose scalar value will be $r_1 \div r_2$ and whose direction angle will be that of z_1 diminished by that of z_2.

Again as in multiplication, it is often convenient to consider the divisor as an operator; and from this viewpoint we may say that the result of operating on a vector z_1 by a vector z_2 as a divisor will be to divide the scalar value r_1 by r_2 and to swing the vector through an angle $-\,\theta_2$.

Multiplication by a vector therefore revolves the vector operated on through a + angle and division by a vector revolves it through a — angle. This is all consistent with the mutually reciprocal relations of multiplication and division. As in algebra, multiplication and division by the same factor will cancel; so here with vectors, multiplication and division by the same vector leave the original vector unchanged.

It will be instructive to deduce the result for the case of division in a somewhat different manner as follows: We have
$$\frac{x_1 + i\,y_1}{x_2 + i\,y_2} = \frac{(x_1 + i\,y_1)\,(x_2 - i\,y_2)}{(x_2 + i\,y_2)\,(x_2 - i\,y_2)} = \frac{(x_1 + i\,y_1)\,(x_2 - i\,y_2)}{x_2^2 + y_2^2} \tag{9.1}$$
In Fig. 16 the two vectors are represented as z_1 and z_2. Likewise the vector $x_2 - i\,y_2$ is represented by OP. Then in the final member of (9.1) the numerator is the product of two vectors z_1 and OP. But

we know that such a product will be a vector having a scalar value equal to the product of the scalars of the two factors. The scalar of z_1 is the length r_1 and the scalar of OP is evidently the length r_2. Hence the scalar of the numerator will be $r_1 r_2$. But the denominator is r_2^2. Hence the final scalar will be $r_1 r_2 \div r_2^2 = r_1 \div r_2$ as before. Again the direction angle of the numerator will be the sum of θ_1 and $-\theta_2$, again the same as before.

From either of these two points of view we can determine the value of the reciprocal of a vector. Thus put

$$\frac{1}{x_1 + i y_1} = x_2 + i y_2$$

where $x_2 + i y_2$ simply indicates the unknown result in its most general form.

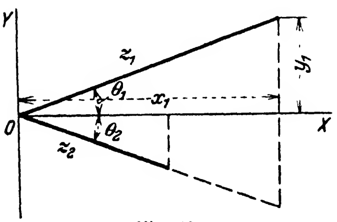

Fig. 17.

Then $\quad 1 = (x_1 + i y_1)(x_2 + i y_2)$

Here 1 is the result of operating on z_1 by some at present unknown vector z_2. From the rules of multiplication we known that scalar $r_1 r_2 = 1$. Hence $r_2 = 1 \div r_1$. The scalar r_2 is therefore the reciprocal of the scalar r_1. Again the direction angle of the product $r_1 r_2 = 1$ is 0. Hence $\theta_1 + \theta_2 = 0$ or $\theta_2 = -\theta_1$. Hence the vector z_2 will have a scalar value $1 \div r_1$ and a direction angle $\theta_2 = -\theta_1$. See Fig. 17.

Or again, following the method indicated in (9.1).

$$\frac{1}{x + i y} = \frac{x - i y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{i y}{x^2 + y^2} \tag{9.2}$$

Vector interpretation of the right hand side will give the same results as before.

We may now develop these same results for multiplication and division by the use of the second or exponential form of a vector.

$$\text{Given} \quad z = re^{i\theta}$$
$$\text{Then} \quad m z = m re^{i\theta}$$

which interprets in exactly the same manner as by the use of the component form, $z = x + i y$.

Again take:

$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2}$$

Then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

The product is, therefore, a vector with scalar value $r_1 r_2$ and angle $(\theta_1 + \theta_2)$, the same result as before.

Again:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$
$$\frac{1}{z} = \frac{1}{r} e^{-i\theta}$$

These interpret exactly the same as by the use of the component form $z = (x + iy)$.

Again:
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$z_2 z_1 = r_2 r_1 e^{i(\theta_2 + \theta_1)}$$

But $r_1 r_2 = r_2 r_1$ and $(\theta_1 + \theta_2) = (\theta_2 + \theta_1)$

Hence
$$z_1 z_2 = z_2 z_1$$

This is readily extended to any number of factors:
$$z_1 z_2 z_3 \ldots\ldots = r_1 r_2 r_3 \ldots\ldots e^{i(\theta_1 + \theta_2 + \theta_3 + \ldots)}$$

And since the values of $r_1 r_2 r_3 \ldots\ldots$ and of $\theta_1 + \theta_2 + \theta_3 \ldots\ldots$ are the same regardless of the order of the terms, it follows that the value of the product of any number of vectors will be the same regardless of the order of the individual factors.

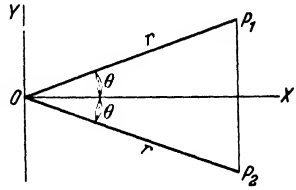

Fig. 18.

These results for multiplication and division are thus reached more simply and directly by the use of the exponential vector form than by the component form. The derivation of the results by the use of both forms, however, has served to illustrate some of the more simple algebraic manipulations to which these forms may be subject, and the double proof of these basic operations is thus justified.

One or two other relations of interest may be developed.

Thus, given the vectors:
$$z_1 = re^{i\theta} \text{ and } z_2 = re^{-i\theta}$$

These are evidently vectors with length r and angle $+\theta$ and $-\theta$ as represented by OP_1 and OP_2, Fig. 18. A change in the sign of the angle θ has, then, the effect of reflecting the vector in the axis of X, but without change otherwise.

Again $re^{i\theta} \times re^{-i\theta} = r^2 e^0 = r^2$. This shows that the product of a vector and its image in the axis of X is a real quantity r^2 laid off along the axis X.

10. Powers and Roots of a Vector. Given the vector
$$z = re^{i\theta}$$

Then
$$z^n = r^n e^{in\theta}$$

That is, the scalar part of z^n will be the scalar of z raised to the ***nth*** power and the angle of z^n will be n times that of z.

This simple relation holds, furthermore, whether n is whole or fractional so that we may put likewise:
$$z^{1/n} = r^{1/n} e^{i\theta/n}$$
$$z^{m/n} = r^{m/n} e^{im\theta/n}$$

Thus the square root of a vector
$$z = re^{i\theta}$$

will have a scalar value of $\sqrt{r}$ and an angle $\theta/2$. And similarly for other values of n.

Since a power is the result of a continued multiplication by the same factor, the rule for powers may be derived directly from that for multiplication, as is obvious.

Similarly and inversely the root used as a continued factor will produce the base number and this relation applied to the rule for multiplication will obviously lead to the same result as found otherwise for roots.

11. Vector Equations of Common Curves. It will be instructive as well as useful to note here the forms which are taken by the vector equations to some of the more common lines and curves in a plane.

(*a*) *Straight Line Parallel to Axis of X.*

$$z = x + ib$$

It is evident that if x be given values from $-\infty$ to $+\infty$ the points given by such a vector will trace a line parallel to X and at a constant distance b above it.

(*b*) *Straight Line Parallel to Axis of Y.*

$$z = a + iy$$

Similarly it is clear that this will give a line parallel to Y and at a constant distance a from it.

(*c*) *The General Straight Line* $y = mx + b$.

In the vector equation $z = x + iy$ substitute as above for y and we have

$z = x + i(mx + b)$ or otherwise if we substitute for x,

$$z = \frac{(y-b)}{m} + iy$$

(*d*) *Any Line with a Polar Equation in the Form* $r = f(\theta)$

$$z = f(\theta)e^{i\theta}$$

This will be clear by remembering that the polar form means a length $r = f(\theta)$ laid off at an angle of θ with the axis of X while the vector form put into words means the same thing.

(*e*) *Circle with Center at the origin and Radius a.*

$$z = ae^{i\theta}$$

The polar form of such an equation is $r = a$ and hence from (*d*) the vector equation will be as above.

(*f*) *Circle with Center at the Point Determined by the Vector* z_0 *and with radius a.*

$$z = z_0 + ae^{i\theta}$$

This is evidently the vector location of any point on the circumference of such a circle.

(*g*) *Any Curve Having an Equation in the General Form* $y = f(x)$

$$z = x + i f(x)$$

This is obvious by substitution in $z = x + i\, y$.

(*h*) *The Parabola.* $y = a x^2$

$$z = x + i a x^2$$

This is merely as an illustration of (*g*).

12. Differentiation of a Vector. Vectors may vary in the same continuous way as ordinary functions and hence should be subject to the operations of the differential calculus.

It must not be forgotten, however, that the geometrical meaning of a differential is not the same with vectors as when dealing with scalar quantities. Thus in the latter case we have

$$x_2 - x_1 = \triangle x$$

That is, $\triangle x$ is the numerical difference between the values x_1 and x_2 of a variable x; or geometrically, it is the difference in linear length between two lengths x_1 and x_2. And then at the limit where this difference becomes very small, we may denote it by dx instead of $\triangle x$.

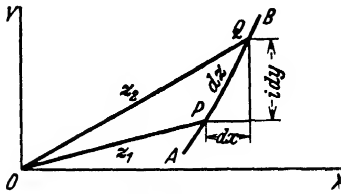

Fig. 19.

On the other hand with vectors, as in Fig. 19, we have $z_2 - z_1 = PQ$ or at the limit, $dz = PQ$. This follows immediately from the subtraction theorem for vectors. Thus if AB is any line or path in the plane XY and P, Q are two points near together on this line, then $d\,z$ is an element of the line or path.

It follows, therefore, that the element of the path lying between the ends of the two vectors z_1 and z_2 (assumed very near together) will represent the differential of the vector at this point.

Again, as with differentiation in the usual case, we may here suppose the total dz made up of two parts dx and idy as shown. Then as a vector equation we have:

$$dz = dx + idy \tag{12.1}$$

This again is consistent with the differentiation of

$$z = x + iy$$

The operation of differentiation may also be developed from the exponential form

$$z = re^{i\theta}$$

Thus: $$dz = ire^{i\theta}\, d\theta + e^{i\theta}\, dr$$

In the first of these two parts, $rd\theta$ is a *length* equal to PR, Fig. 20. Then $rd\theta e^{i\theta}$ is a vector of this length directed along OP, (PR_1). This

multiplied by i turns it through 90^0 or into PR. The first part thus represents the vector PR.

The second part is evidently a vector of the length dr at an angle θ with X. This is represented by RQ. The two together give therefore the sum of the two vectors $PR + RQ = PQ$ or

$$dz = PQ$$

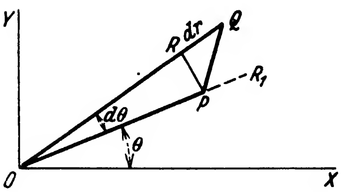

Fig. 20.

In the following chapters will be found some further development of certain properties of the complex variable $(x + iy)$ in its relation to vectors and vector fields.

CHAPTER VI

VECTOR FIELDS

1. Introductory. Picture a space, three-dimensional in the general case, throughout which, at every point, some physical characteristic has a definite magnitude and direction. Such a space with reference to the distribution of the vector representing such physical magnitude may be termed a "*vector field*". Thus if we imagine an indefinite space, void except for a single sphere of matter at its center, then the distribution of gravity throughout such space will constitute a vector field. At every point the direction will lie toward the center of the sphere and the magnitude will vary inversely as the square of the distance from this center. At every point in such a space, therefore, there could be drawn a vector representing the force of gravity in direction and magnitude. If instead of one body there are two or more, the same general conditions will hold, but the field will be more complex in character. Nevertheless at each and every point the force of gravity will have a single definite value and a single definite direction, and will constitute a vector field.

A vector field in general is single-valued; that is at any one point the vector will have a single value and a single direction.

Again picture a fluid medium of indefinite extent moving through a space in which are anchored solid bodies as obstructions and around which the fluid must flow. In such a space, assuming a steady condition of movement, there will be, in general, at every point, a single definite value of the velocity, both as to magnitude and direction. The distribution of the velocity of flow throughout such a space constitutes, therefore, a vector field[1].

Velocity vector fields play an important part in many problems in the domain of aerodynamics, and a study of the more important

[1] The existence of certain singular points where this single-valued condition is not fulfilled will be noted in later chapters.

properties of such fields will form the subject matter of the present and following chapters. For the development of these properties two general methods are employed.

(1) The development of various properties and relations through the use of the analytical geometry of three or of two dimensions.

(2) The development of various properties and relations, especially in two-dimensional fields, through the use of the complex variable $(x + iy)$.

We shall develop the early part of the subject in the order as stated.

2. Vector Components. In Fig. 21 let P be any point in space referred to coordinate axes X, Y, Z, with origin at O. Let PA denote the length and direction of a vector at P, and let α, β, γ denote the angles between PA and the directions of the axes X, Y, Z, respectively. Then putting V for the value of the vector and u, v, w, for its three components along the directions of X, Y, Z, we shall have

$$\left.\begin{aligned} u &= V\cos\alpha \\ v &= V\cos\beta \\ w &= V\cos\gamma \end{aligned}\right| \qquad (2.1)$$

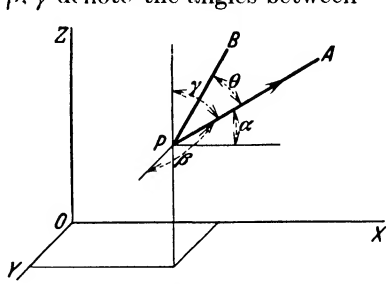

Fig. 21.

In accordance with the well known theorems of kinematics and mechanics we may take a vector as fully represented by its components and inversely the three components conjointly as fully represented by the resultant vector. In other words the vector and its components are mutually equivalent. The vector is the resultant of its components and the components are the individual directional effects of the resultant. Also we know that the resultant of three components u, v, w, may be expressed in two ways: $V = \sqrt{u^2 + v^2 + w^2}$ (2.2)

and $$V = u\cos\alpha + v\cos\beta + w\cos\gamma \qquad (2.3)$$

Hence if we put for u, v, and w as above we shall have:

$$V = V\cos^2\alpha + V\cos^2\beta + V\cos^2\gamma$$

or $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (2.4)$$

Again suppose at P that we have another line PB inclined at angles λ, μ, ν to the X, Y, Z, directions. Denote the angle between PB and the vector PA by θ.

Then for the component of V along PB we have

$$V_{PB} = V\cos\theta$$

But remembering that V may always be replaced by its components u, v, w, we shall have for V_{PB} the sum of the components of u, v, and w, along PB. Hence

$$V_{PB} = u\cos\lambda + v\cos\mu + w\cos\nu \qquad (2.5)$$

Putting for u, v, and w their values as components of V, we then have: $V_{PB} = V \cos\theta = V \cos\alpha \cos\lambda + V \cos\beta \cos\mu + V \cos\gamma \cos\nu$

or $$\cos\theta = \cos\alpha \cos\lambda + \cos\beta \cos\mu + \cos\gamma \cos\nu$$

a well know result in the analytical geometry of three dimensions.

In a two-dimensional space, taking θ for the inclination with X, (2.1) becomes

$$\left.\begin{aligned} u &= V \cos\theta \\ v &= V \sin\theta \end{aligned}\right\} \tag{2.6}$$

whence $$V = \sqrt{u^2 + v^2} \tag{2.7}$$

and (2.3) reduces to

$$V = u \cos\theta + v \sin\theta \tag{2.8}$$

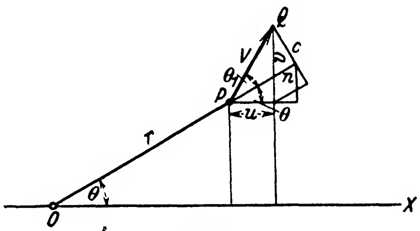

Fig. 22.

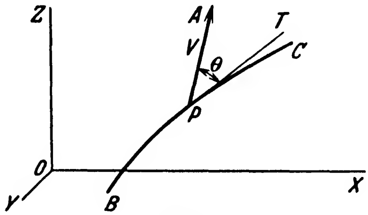

Fig. 23.

In polar coordinates as in Fig. 22 let now θ denote the angle of the radius vector to a point P and θ_1 the angle between the vector $PQ = V$ and the radius vector. Then let

n = value of vector component along radius vector r

c = value of vector component along $\perp$ to radius vector r

Then obviously
$$\left.\begin{aligned} u &= n \cos\theta - c \sin\theta \\ v &= n \sin\theta + c \cos\theta \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} n &= u \cos\theta + v \sin\theta \\ c &= v \cos\theta - u \sin\theta \end{aligned}\right\} \tag{2.10}$$

$$V = n \cos\theta_1 + c \sin\theta_1 = u \cos(\theta + \theta_1) + v \sin(\theta + \theta_1) \tag{2.11}$$

$$V = \sqrt{n^2 + c^2} \tag{2.12}$$

3. Line Integral. In Fig. 23 let BC be a curve in space referred to axes X, Y, Z, and P any point on the curve. Let this be in a vector field and let V denote the vector at P and PT the tangent to the curves Let θ denote the angle between V and the tangent. Then $V \cos\theta$ is the component of V along the line of the curve at P. Let ds denote an element of length at P. Then the product of the component $V \cos\theta$ by the length of the element ds is found to be of peculiar significance in the examination of the properties of vector fields and the summation of such elements along a line or path, as from B to C, has received the

name of *line integral*. We thus speak of the line integral of the vector V along the path BC. Putting L for *Line Integral* we have then,

$$L = \int_B^C V \cos \theta \, ds \tag{3.1}$$

But the component $V \cos \theta$ may also be expressed as the sum of the components of u, v, and w taken along the direction of the tangent PT, where, as before, u, v, and w are the rectangular components of V, see (2.5). If then, λ, μ, ν are the direction angles of the tangent PT, we shall have, as an alternative expression for the line integral

$$L = \int_B^C (u \cos \lambda + v \cos \mu + w \cos \nu) \, ds$$

But $ds \cos \lambda = dx$, $ds \cos \mu = dy$, and $ds \cos \nu = dz$. Hence again,

$$L = \int_B^C u\,dx + v\,dy + w\,dz \tag{3.2}$$

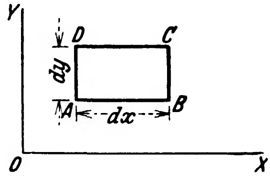

Fig. 24.

Here, of course, u, v, and w refer to the vector and dx, dy, dz, to the element ds of the path BC.

We thus have two expressions (3.1) and (3.2) for the line integral of a vector along a path, either of which may be used according to convenience or purpose.

4. Line Integral in Two Dimensions. In a field of two dimensions, (3.1) remains the same in form while (3.2) reduces to the form

$$L = \int_A^B (u\,dx + v\,dy) \tag{4.1}$$

We have now to develop an expression for L taken around a small element of area $ABCD$, Fig. 24. We take the dimensions of the area dx and dy as shown. Taking then the integral in the cyclical order $ABCDA$, the first element will be $u\,dx$. But immediately the question arises as to the value of u. In a field of continuously varying vector strength, we cannot take u as constant from A to B. We can, however, define the u for AB as the value for the midpoint. Whatever the variation of u over AB, the error in the product $u\,dx$ will then be a small quantity of the second order and hence negligible. In the same way we take v as the value at the midpoint of AD. Now assuming u and v to vary along X and Y, the corresponding values at the midpoints of DC and BC will be respectively:

$$u + \frac{\partial u}{\partial y} dy, \quad v + \frac{\partial v}{\partial x} dx$$

With these velocity components the line integral around $ABCDA$ becomes,

$$u\,dx + \left(v + \frac{\partial v}{\partial x} dx\right) dy + \left(u + \frac{\partial u}{\partial y} dy\right) (-dx) + v\,(-dy)$$

This gives:
$$dL = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy \tag{4.2}$$

If v and u are constant, dL becomes zero and the line integral around the element vanishes.

In (4.2) $dx\, dy$ is the area of the element. Denote this by dS. Then we have
$$\frac{dL}{dS} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Passing toward the limit, we may conceive a small element of area of any form whatever, as built up of still smaller rectangles and hence in the usual manner in dealing with such problems, we readily extend the above result to an area of any form, provided it is so small that the values of $\partial v/\partial x$ and $\partial u/\partial y$ do not sensibly vary over its extent. For any such small area, therefore,

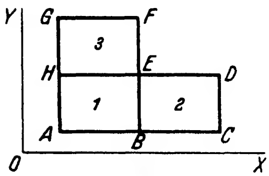

Fig. 25.

$$dL = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dS$$

This reduces the value of the line integral around the boundary to expression in terms of area.

Again it is obvious that for any element of length ds, the line integral due to an X or a Y component alone will be
$$d\,L_u = u \cos\theta\, d\,s = u\, d\,x$$
or
$$d\,L_v = v \sin\theta\, d\,s = v\, d\,y$$
and hence for any line straight or curved
$$L_u = \int u \cos\theta\, d\,s = \int u\, d\,x$$
or
$$L_v = \int v \sin\theta\, d\,s = \int v\, d\,y$$

Again let Fig. 25 represent three elements of area, each dx by dy, as indicated by the three rectangles. Then we have to show that the L around the outside boundary $ACDEFGA$ will be equal to the sum of the L's about the three elements of which the whole is composed.

Let us assume the L's for the three elements taken individually all in the same cyclical order, counter-clockwise. Then L for (1) will contain the value for BE in the direction $B \ldots E$, while L for (2) will contain the value for the same line but in the direction $E \ldots . B$. These two values will therefore cancel in the sum. In the same way it follows that the values for EH and HE will cancel in the sum of the L's for elements (1) and (3). In general it is clear that in taking the line integrals of the several elements of a complex area, any line which forms the boundary between two adjacent elements will be traversed twice in opposite directions and hence these values will cancel out in the sum. Hence in the final summation, only those elements will be represented which lie on the outer boundary of the area and these will all be traversed in the same cyclical direction and hence the results will sum together

and give exactly the same as though the inner elemental paths were omitted and the traverse were made directly around the outer boundary.

Thus in Fig. 25 the sum of the L's for elements (1), (2) and (3) taken individually will be the same as will result from a direct traverse of the outer boundary $ACDEFGA$.

The extension from a broken line to a curved line boundary is made in the usual way, by assuming the elements small to the limit, and thus as giving a boundary differing less than any assignable quantity from the given curved boundary.

We thus reach the interesting result that the line integral of a vector around any closed curve may be expressed in two forms as follows: First from (4.1)

$$L = \int (u\,d\,x + v\,d\,y) \tag{4.3}$$

where the integration is extended around the outer boundary; then from (4.2)

$$L = \int\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d\,x\,d\,y \tag{4.4}$$

where the integration is extended over the area itself.

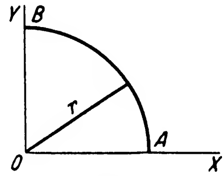

Fig. 26.

By way of a simple illustration of the identity of (4.3) and (4.4) assume a vector field with the components

$$u = a\,y^2$$
$$v = b\,x^2$$

and let us find the line integral around the quadrant OAB of a circle with radius r as in Fig. 26.

The equation to the circle is

$$x^2 + y^2 = r^2$$

Taking first (4.3)

$$L = \int_r^0 a\,y^2\,d\,x + \int_0^r b\,x^2\,d\,y \tag{4.5}$$

Next taking (4.4)

$$L = 2\,b \int_0^r\int_0^x x\,d\,x\,d\,y - 2\,a \int_0^r\int_0^y y\,d\,x\,d\,y$$

In the first integral, integrating first for x, the limits will be 0 and x and those for y, 0 and r, while in the second integral, integrating first for y, the limits for y will be 0 and y and then for x, 0 and r. Hence,

$$L = b \int_0^r x^2\,d\,y - a \int_0^r y^2\,d\,x \tag{4.6}$$

Comparing this with (4.5) it is seen that the first integral of (4.6) is the same as the second of (4.5) and with the limits the same while the second of (4.6) is the same as the first of (4.5) but with the limits reversed. Hence the minus sign of the second term in (4.6) taken with the reversal of the limits make these two the same and therefore the two equations will give the same result.

We have thus established the three following important results:

(1) The line integral around an element of area in the field XY is given by
$$dL = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d x\, d y$$

(2) The integration of this expression over a given area will give the summation of these individual line integrals and this, by the mutual cancellation of all "internal records", will give the line integral along the boundary of the area as indicated by the equation:
$$L = \int (u\, d x + v\, d y) = \int\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d x\, d y \tag{4.9}$$

(3) The condition that $L = 0$ for a given closed boundary is evidently:

(a)
$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \tag{4.10}$$

Such condition to prevail over the area enclosed by the boundary.

Again, as a special case:

(b) $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$, or u and v constant and hence V constant.

The condition of Eq. (4.10) will be found to determine a very important type of fluid motion to which attention will be given at a later point.

The line integral of a vector around a closed boundary is often called the circulation of the vector. This term is very commonly employed where the vector represents a velocity as in a field of fluid flow. The circulation around any closed path is then the line integral of the velocity taken around such path.

Kelvin's Theorem. In advanced treatments of the subject of fluid mechanics[1] proofs of Kelvin's Theorem are given. In accordance with this theorem the circulation along a given fluid line remains constant with time. A fluid line is defined as a line always composed of the same fluid particles. As a consequence if the circulation about any closed circuit is zero, it will always remain zero. That is, if the condition of zero rotation for irrotational motion is once fulfilled, it will always remain so. Or again, in an ideal fluid without rotation, rotation can never arise. Compare this with the statement in Division B III 1 regarding rotation when once this condition is assumed to exist. This all means simply that the properties of the ideal fluid do not change with time and may be accepted without more formal proof.

Simply and Multiply Connected Regions. A simply connected region in space is one in which any closed circuit can be contracted to a point without cutting any boundary of the region. Or otherwise expressed, it is one in which any circuit APB between two points A and B can

[1] LAMB, H., "Hydrodynamics", 5th ed., p. 34, Cambridge, 1924.

be continuously varied to a different path AQB without cutting any boundary of the region. Thus the space inside a sphere is a simply connected region. The space inside a torus or anchor ring is not. Two circuits such as APB and AQB above are called reconcilable. In a multiply connected region it is possible to draw a certain number of paths which are not reconcilable one with another. Thus for the torus there are two such paths and the region is said to be doubly connected.

It will be clear that in a simply connected region with irrotational motion, (see 8) the line integral around a circuit $APBQA$ will be zero and hence the integral along any two circuits APB and AQB will be equal and of the same sign. In multiply connected regions these simple relations no longer hold. The problems arising in aerodynamics may, for the most part, be treated without involving the properties of multiply connected regions and no further development of the subject will be here undertaken. The interested reader may consult more advanced treatments of the subject [1].

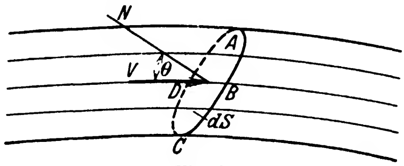

Fig. 27.

5. Vector Flux. Assume a vector field as in **1**. This may be better visualized by assuming the vector to represent the velocity of flow in the movement of a fluid. There will be, then, for each point in the field, a vector representing at that point, the direction of flow and the velocity in that direction. If, then, starting from any point we should follow in the direction of flow from one point to another, we shall trace out a continuous line or path of flow such that at each point the direction of flow will lie tangent to the path. In the case of fluid flow, such a path is called a stream-line. It is clear that the entire field may be pictured as filled with these lines of flow, thus giving a complete representation of the direction of flow at each point and throughout the field. Quite independent, however, of the physical character of the vector, such lines may be mapped out in any vector field and may be considered as representing lines of vector action or of vector flux.

Now picture, in space, any small area with its closed boundary, see $ABCD$, Fig. 27. Then, following the picture of fluid flow, there will be a stream-line touching each point of the boundary $ABCD$ and the collection of such lines will form the surface of a tube or channel of flow. This is called a tube of flow or tube of flux in the more general sense.

If such a tube is small to the limit, the area $ABCD$ will be sensibly a plane and the value of the vector will not vary sensibly over this area. Such a small tube of vector flux is sometimes called a vector filament.

[1] Lamb, H., "Hydrodynamics", 5th ed., p. 47, Cambridge, 1924.

Let $V =$ the scalar value of the vector (velocity in a field of fluid flow). Or otherwise V may be assumed as the value at the center of the small area dS, in which case it may be extended over the area dS with an error vanishingly small at the limit.

$dS =$ area of the small surface $ABCD$.

$dA =$ the cross section area of the tube at dS.

N denote the normal to dS.

$\theta =$ the angle between V and N, or its equal, the angle between the surface dS and the surface dA.

Then from geometry we have $dA = dS \cos \theta$ and multiplying by V,

$$VdA = V\,(dS \cos \theta) = V \cos \theta \, dS \tag{5.1}$$

The product VdA is the rate of fluid flow across the area of the tube at right angles to the direction of the flow. This again is equal to the product of the vector value V by the projection of the surface dS on a plane normal to the direction of flow, or again to the product of the component of V along a normal to dS by the area dS. It results that the flow through or across any surface extending as a geometrical partition across a tube of flow will be the same independent of the inclination of the surface to the direction of flow and will be measured as in (5.1). We have thus shown that if a small tube or filament of flow is cut by a surface dS at a point where the value of the velocity is V and the inclination of dS to the cross-section of the tube is θ, then the flow across the surface will be measured as in (5.1). It is evident, however, that where the lines of flow are straight and parallel, with V uniform in value and the surface a plane, the formula will hold regardless of the size of the tube or conduit of flow.

In the case of any fluid field in general, however, the measure of the flow across any surface of any size and form, will require an integration over the surface, of a differential expression similar to (5.1), thus giving

$$\overline{V}A = \int V \cos \theta \, dS \tag{5.2}$$

where $\overline{V}$ might be called the average velocity of the vector over the area A.

While a field of fluid flow has been used to illustrate the significance of (5.1), (5.2), it is clear that these equations are true for any vector field regardless of its physical nature, and that statements parallel to those regarding fluid flow may be made for what may be termed more generally, *vector flux* or the flux of a vector across or through a surface.

6. Vector Flux through a Volume. Let the diagram, Fig. 28, represent an element of volume bounded by the edges dx, dy, dz, and referred to axes X, Y, Z. Let this be in the field of a vector V with components u, v, and w as hitherto.

Let u be the value of the X component of V at the center point of the face $dy\,dz$ nearer the plane YZ. Then the value at the corresponding point in the opposite face $dy\,dz$ will be

$$u + \frac{\partial u}{\partial x} dx$$

Counting the flux through the first face inward as positive and that through the second face outward as negative, we shall have the difference representing a net flux outward, measured by

$$-\frac{\partial u}{\partial x} dx\,dy\,dz$$

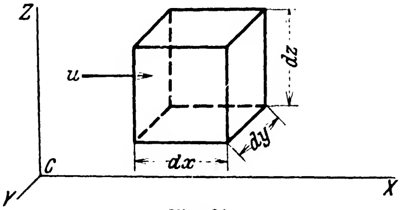

Fig. 28.

with similar expressions for the net flux for the components v and w through the faces $dxdz$ and $dxdy$ respectively. The expression for the net flux through the element as a whole and of volume $dx\,dy\,dz$ will be therefore:

$$-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\,dy\,dz \tag{6.1}$$

The expression within the parenthesis may be viewed as a coefficient of concentration, or otherwise as a measure of the concentration per unit volume. A negative sign, as above, implies a relative depletion (negative concentration) within the volume—an excess of outflow over inflow, while a positive sign implies positive concentration, an excess of inflow over outflow. This expression is termed the *divergence* of the vector V.

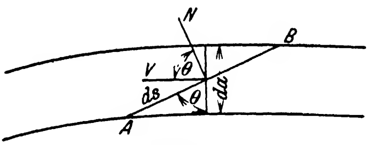

Fig. 29.

A special distribution of the vector field will cause this expression to vanish, thus implying neither concentration nor depletion within the volume. The condition for this special distribution is therefore:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6.2}$$

implying a movement in which there is no change in the mass of fluid within the element, hence no change of density and hence movement as of an *incompressible* fluid.

7. Vector Flux in Two Dimensions. If we are dealing with a plane vector, the field becomes reduced to a plane and the tube of flow to a thin sheet of flow bounded by two stream-lines as in Fig. 29. In this case (5.1) will become

$$V\,da = V\,(ds \cos \theta) = V \cos \theta\, ds \tag{7.1}$$

where

da = the breadth of the band of flow in a direction $\perp$ to the direction of flow.

ds = length of line AB = element of a path cutting the band of flow.

θ = the angle between da and ds or its equal, that between V and the normal to ds.

Certain details of importance in connection with the study of the problems of a two-dimensional vector field may be further noted. Let Fig. 30 represent a portion of a small band of fluid flow, in a two-dimensional field. The entire region comprised by the figure is supposed to be of such size that V is sensibly constant in direction and value throughout. AC is any line drawn across the band between the lines of flow RR and SS, while AB and CB are any other two lines drawn from A and C and meeting at B. Then the line of flow TT, through B, will divide the original band in two, one lying between SS and TT and one between TT and RR.

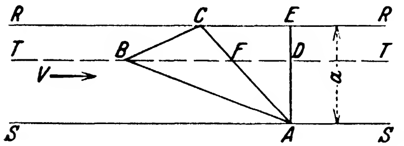

Fig. 30.

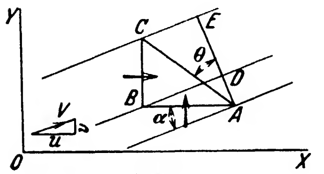

Fig. 31.

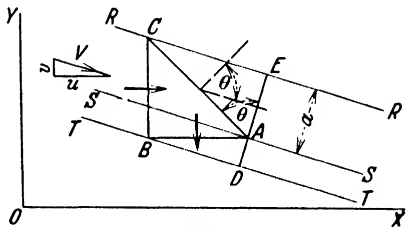

Fig. 32.

Denote by F_{AB} the flow across AB and similarly for other subscripts. Then from the preceding results we shall have:

$$F_{AB} = F_{AF} = F_{AD}$$

$$F_{BC} = F_{FC} = F_{DE}$$

Then by addition: $\quad F_{AB} + F_{BC} = F_{AC} = F_{AE} = Va$

In Fig. 31 we have the case of a vector flow V referred to axes X and Y. Let u and v denote respectively the X and Y components of V. Then $F_{BC} = X$ *component of* V *multiplied into* $BC = u \cdot BC$ and $F_{AB} = Y$ *component of* V *multiplied into* $AB = v \cdot AB$. Then we shall have $\quad F_{AC} = u \cdot BC + v \cdot AB = V \cdot AE = V \cdot AC \cos \theta$

Or if AC denotes an element of a path ds, AB and BC will be represented by $-dx$ and dy and we shall have

$$u\,dy - v\,dx = V ds \cos \theta \qquad (7.2)$$

As a second case, in Fig. 32, RR and SS are again the boundary lines of flow for the line AC representing the element ds of a path across the field. Then as before we shall have:

$$F_{AB} = -v \cdot AB = V \cdot AD$$

$$F_{BC} = u \cdot BC = V \cdot DE$$

Then by algebraic addition.

$$F_{BC} - F_{AB} = u \cdot BC - (-v \cdot AB) = V \cdot AE$$

or $$u\,dy - (-v)(-dx) = u\,dy - v\,dx = V \cdot AE = V \cdot ds \cos \theta$$

In this method of measuring the flow across an element ds of a line in a field of flow, care must be given to the signs of the elements u, v, dx, dy, as they vary according to the different combinations which may arise. Thus in Fig. 31, starting from A, we substitute the path $AB \ldots . BC$ for AC. Then along AB, v is (+) and dx is (—) while along BC, u and dy are both (+). Again in Fig. 32, along AB, v and dx are both (—) and along BC, u and dy are both (+). In the various other possible combinations, the same principles must apply.

In Figs. 31, 32, the signs of dx and dy have been determined on the assumption of a positive ds in the direction AC and the flow V directed as indicated. If we should take positive ds in the direction CA, the signs of dx and dy will be reversed and we shall have

$$\text{Flow} = v\,dx - u\,dy$$

The same reversal of sign will result if we reverse the direction of the flow V. If both are reversed, the sign remains the same.

In any event, and this is important, the expression $(u\,dy - v\,dx)$ is a quantitative measure of the flow across any element ds of a path in the field of two-dimensional flow—the ultimate sign of such flow to be determined by the conditions of the problem.

In order to aid in forming a picture of the physical meaning of these various relations, they have been developed in terms of the language of fluid flow. It will be clear, however, that the results are quite general and hold for vector flux in general, and independent of the physical character of the vector.

8. Rotational and Irrotational Motion. The vector fields which find application in the problems of aerodynamics are naturally those which represent fields of fluid motion. It is shown in kinematics that the motion of a body may be resolved into two components, *translation* and *rotation*. In a motion of translation by itself, the angular relation of the body with reference to a set of axes does not change, but the body as a whole, or any point therein taken as a reference point, may describe any path in space or in a plane, as the motion is in three- or two-dimensions. We take first the case of two-dimensional motion.

Thus in Fig. 33 we have a square the center of which moves along the line AB, while, at the same time preserving a constant angular relation to the axes X and Y. Such a motion is one of translation only.

On the other hand in a motion of pure rotation, the body changes its angular relation to a set of reference axes without change of location as a whole. Thus in Fig. 34 a movement carrying the square from the full line to the dotted line position is, for the square as a whole, one of pure rotation. It should be observed, however, that for a small square lying in the corner of the larger one, as indicated at B, the motion comprises both translation and rotation. Again for a small circle at the center of rotation, as shown, the motion for the circle as a whole is one of rotation, while for a circular element of the same size at P or for a small element of any form, as at Q, the motion is one of translation combined with rotation. It follows that only a vanishingly small element of a plane such as $ABCD$ can have a motion of pure rotation, and that only when it is at the center of rotation for the plane. In general then if we take the ultimately small elements of any body moving in a plane, the motion of each element will be composed of translation combined with rotation, the only exception being the vanishingly small element at which, for the moment, the center of rotation is located. These results, as based on the indications of Fig. 34, obviously apply to the motion of any solid body in two-dimensional motion, or to any body or mass moving as such.

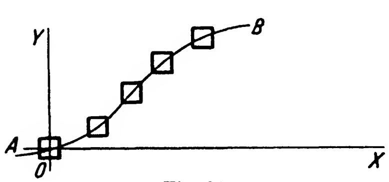

Fig. 33.

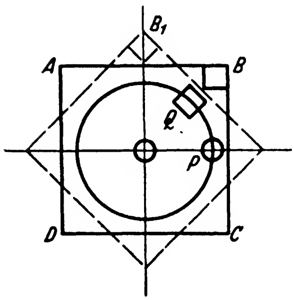

Fig. 34.

In the case of a fluid mass, however, the small elements of the mass do not necessarily maintain their angular position relative to the mass as a whole, or in other words, a small element may move in any manner whatever and independent of the movement of the other elements making up the fluid mass of which it is a part. For this reason it is not possible to give to the terms translation and rotation as applied to a fluid mass as a whole, the same precision as in the case of a solid body. With reference to fluid motion, therefore, we must understand these terms as applying only to the movement of the ultimately small elements of the fluid mass as a whole. The terms *irrotational* and *rotational* are applied to the two types of fluid motion which may thus result. In the former, we have no rotation of the ultimately small element. This

gives a motion of translation only. In the latter we have, in general, a motion of translation combined with one of rotation.

Before taking up the subject proper of rotational or irrotational movement in fluids, we shall find it convenient to establish one or two simple propositions in kinematics.

In Fig. 35 let AB represent any line parallel to Y and carried rigidly by radii OP and OQ. Let this assembly be given a motion of rotation about O as center. Then at the instant shown in the diagram, any point in the line, as P, R, S, or Q, will be moving in the arc of a circle with OP, OR, OS, or OQ, as radius, in a direction $\perp$ to such radius and with linear velocity measured by the product of the radius and the angular velocity.

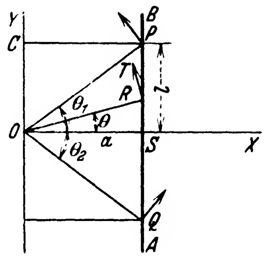

Fig. 35.

We have now to prove that the component of such velocity along the direction of the line itself is independent of the location of the point on the line.

Let a = distance OS

ω = angular velocity

Then, considering the point R, the linear velocity in the direction RT is $OR\,\omega$. The component of this along AB is $OR \cos\theta\,\omega$. But $OR \cos\theta = a$. Hence the component linear velocity of any point R along the direction of the line AB is $a\,\omega$ which is independent of θ or of the location of the point. This is obviously the velocity of the point S and, relative to the velocity at this point, it is seen that what is lost in the ratio of the cosine by taking a component at an angle θ is gained in the ratio of the secant by the increase in the length of the radius OR.

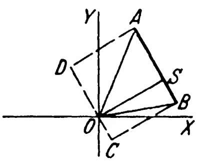

Fig. 36.

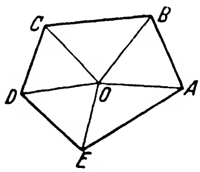

Fig. 37.

The line integral of this velocity along any length l of the line AB, will be, therefore, measured by $a\,l\,\omega$. But $a\,l$ is the area between the line l and the axis Y and this is twice the area of the triangle OSP. This result is readily generalized to a line AB Fig. 36, considered as carried by two radii OA and OB and rotating about O as center. In fact it is readily seen that since the line is rotating about O, we may take axes momentarily parallel and $\perp$ to AB, in which case the demonstration above holds. It thus follows that for any line AB rotating about a center O with angular velocity ω, the line integral of the velocity along the length of the line is measured by twice the area of the triangle OAB multiplied by the angular velocity ω.

If this theorem is extended to an assemblage of lines forming a closed contour as in Fig. 37, it is evident that for the polygon in rotation about O, the line integral about the contour will be measured by the product of the angular velocity ω into twice the area of the polygon $ABCDE$.

The same relation may be shown to hold independent of the location of the center of rotation. Thus in Fig. 38a let $ABCDE$ be any irregular polygon in rotation about a center O, with angular velocity ω. Then applying the theorem above to the total contour, it will be readily seen that, with a continuous cyclic direction around the contour, the area $OABCDO$ will be counted positive while the area $OAEDO$ will be counted negative. The difference will be the area of the polygon and it will therefore result, the same as before, that the line integral around the contour $ABCDE$ will be measured by the product of the angular velocity ω by twice the area of the polygon.

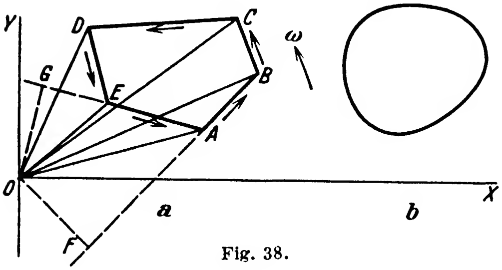

Fig. 38.

Then in the usual manner we may pass from a many sided polygon to a curved contour, Fig. 38b and thus show that the relation is true for any figure of any form whatever. In the case of a circle rotating about its center, the relation comes directly in the form

$$Line\ integral = r\,\omega \times 2\,\pi\,r = 2\,\pi\,r^2\omega = 2\,\omega \times area$$

In developing these theorems we have assumed a rigid plane contour rotating about an axis $\perp$ to its plane. So far as relative motions are concerned, it will evidently be the same if we assume the contour fixed with a separate plane rotating about the contour. The preceding theorems would then apply to the measure of the line integral of the motion in the rotating plane taken along the line of the contour.

Returning now to the subject of rotation, this term is defined as follows.

The rotation of a plane figure relative to a field of motion represented by a plane in rotation about a fixed center, is measured by the line integral of the motion, carried around the contour of the figure, divided by its area or: *Rotation* $=$ *Line integral* $\div$ *area.*

However in a field of fluid motion, it cannot be assumed that anything more than a very small element of the fluid is moving with pure rotation and in such a field the theorems above and the definition of rotation can only be assumed to apply to an element of the fluid indefinitely small at the limit. We therefore, for our present purposes,

define rotation as the line integral of the motion taken around a vanishingly small element of area, divided by the area.

But, as we have seen above, the line integral in any such case is measured by twice the product of the area of the element by the angular velocity. Hence the measure of rotation reduces to the simple form of twice the angular velocity or:

$$\textit{Rotation} = \textit{Line Integral} \div \textit{Area} = 2\,\omega$$

In (4.2) we have the analytical expression for the line integral of a vector about a small element of area, see Fig. 39:

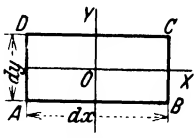

Fig. 39.

$$\textit{Line Integral} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \qquad (8.1)$$

Hence $$\textit{Rotation} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (8.2)$$

With the line integral taken in the cyclical order $ADCBA$, the result would be the same as above but with the sign reversed:

$$\textit{Rotation} = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

It will be noted that these expressions refer to a particular point in the field and that as the value may vary from point to point over the field, so will the value of the rotation change.

It follows that the condition for irrotational motion is simply that the integration of (8.1) over the field shall vanish. But this means that everywhere (8.2) must be zero, and this taken with (4.2) means that the line integral of the velocity taken around any closed path in the field must be zero.

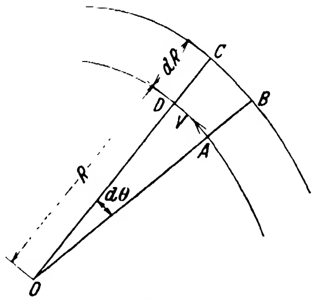

Fig. 40.

The analytical condition for this is obviously:

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \text{ as in } (4.10)$$

Rotation in a Field Defined with Reference to a Center of Curvature. While the discussion of rotation in terms of rectangular coordinates is entirely general, the form taken by the expression for rotation where the element is defined with reference to a center of curvature of the lines of flow, is of special interest and will find application in certain problems in fluid mechanics.

In Fig. 40 let AD and BC denote small arcs of two adjacent lines of flow, O the center of curvature and OB, OC two radii thus forming the small element $ABCD$ of the field of flow. Then if V denotes the

velocity along AD, and counting dR positive outwards, the circulation around $ABCD$ will have the value,

$$(R + dR)\, d\theta\, (V + \frac{\partial V}{\partial R} dR) - R d\theta\, V = V d\theta\, dR + (R + dR) d\theta\, dV$$

But by definition, the rotation, or twice the angular velocity ω of the element, is measured by the circulation around the small element divided by its area. But the area of the element is $R d\theta\, dR$ and dividing by this expression we find,

$$2\omega = \text{rotation} = \frac{V}{R} + \left(\frac{R + dR}{R}\right) \frac{\partial V}{\partial R}$$

But at the limit $(R + dR)/R$ approaches unity and thus, for an indefinitely small element of the field,

$$2\omega = \text{rotation} = \frac{V}{R} + \frac{\partial V}{\partial R}$$

Furthermore since R lies along the line of the normal, we may (taking dN positive outward) write equally well,

$$2\omega = \text{rotation} = \frac{V}{R} + \frac{\partial V}{\partial N} \tag{8.3}$$

9. Rotational and Irrotational Motion in Three Dimensions. In the general case of motion in space, the movement may, as in 8, be decomposed into two parts, one of translation and one of rotation. The movement of translation will have the three component velocities u, v, w. The movement of rotation will have, in the general case, three component rotations about axes parallel to the three axes X, Y, Z, or in planes parallel respectively to YZ, ZX, and XY.

From (8.2) we may write expressions for these three component rotations as follows:

$$\left.\begin{aligned} \gamma_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ \gamma_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \gamma_x &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \end{aligned}\right\} \tag{9.1}$$

For irrotational motion in space, therefore, these expressions individually must all vanish. The relation of this condition to the value of the line integral of the velocity around any closed path in the space field will be established by Stokes' theorem, see IX 3. Anticipating the results of this theorem, we may here state the conclusion that, for irrotational motion in space, the necessary and sufficient condition is that the line integral of the velocity taken about any closed curve in the space field must vanish.

The three expressions in (9.1) are often known as the components of the *curl* of the vector of which u, v and w are the axial components. Thus *Curl* and *Rotation* are two different names for the same characteristics of the vector.

CHAPTER VII
POTENTIAL

1. Potential. In the preceding sections we have investigated certain properties of a vector field, the latter being determined by the three vector components u, v, and w in a three-dimensional field or u and v for a field of two dimensions We have now to examine a special type of field in which the components u, v, and w admit of expression in the following form.

$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} \\ v &= \frac{\partial \varphi}{\partial y} \\ w &= \frac{\partial \varphi}{\partial z} \end{aligned}\right\} \tag{1.1}$$

Or in a field of two dimensions and using u and v as before, we should have

$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} \\ v &= \frac{\partial \varphi}{\partial y} \end{aligned}\right\} \tag{1.2}$$

In these expressions φ denotes a function of the coordinates x, y, and z, such that the partial derivative with reference to x (for example) evaluated for any point in space, will give the value of the component u (in the X direction) at that point, and similarly for the components in the directions of the axes Y and Z.

A vector field in which these conditions are fulfilled is said to have a *Potential* and φ is called the "Potential" of the field or of the vector V.

We must now stop to note in particular the significance of a partial derivative such as $\partial\varphi/\partial x$. It denotes the rate of change of φ relative to x when x alone of the three variables x, y, z, is subject to change, y and z remaining constant. Geometrically it denotes the change in φ relative to a small change in x, when moving anywhere in space in a direction parallel to the axis X. In the case of a space field we may imagine a function φ having everywhere a specific value, but changing from point to point. Then if we imagine a small movement $\triangle x$ in a direction parallel to the axis X, there will result a small change in φ measured by $\triangle \varphi$. Then at the limit when $\triangle x$ becomes vanishingly small, the ratio $\triangle\varphi/\triangle x$ becomes $\partial\varphi/\partial x$, the rate of change of φ relative to x for change in the direction of x; or in brief, the partial derivative of φ along X. In the same manner, of course, we interpret $\partial\varphi/\partial y$, $\partial\varphi/\partial z$.

It is sometimes convenient to denote a line in space by a single letter such as l or n. Such a line may have any direction relative to the three axes X, Y, Z, as determined by the three direction angles α, β, γ. See Fig. 21. Then in exactly the same manner as the movement or change in the directions X, Y, Z, we may use the notation $\partial\varphi/\partial l$ to denote the rate of change of φ along the line l, and similarly for

$\partial\varphi/\partial n$. In general, then, it is to be understood that any expression in the form $\partial\varphi/\partial l$, where φ is a space function of x, y, z, denotes simply the rate of change of φ in space along the direction of this particular line. Thus, if at a given point, the change in φ along the direction of a line l is at the rate of 5 units of φ per unit change in distance along l, then $\partial\varphi/\partial l = 5$. This simple mental picture of the meaning of a space partial derivative will be of value in the study of the properties of vector fields.

In general, the expression for the function φ will not be given in terms of l, but only in terms of x, y, z. Hence so far as the process of differentiation is concerned, we cannot find directly $\partial\varphi/\partial l$, but only $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$. We must, therefore find the relation between $\partial\varphi/\partial l$ and these partial derivatives along the axes X, Y, Z. Picture a point in space with a line l inclined at angles α, β, γ to X, Y, Z. See Fig. 21. Then in accordance with our notation, a change or movement of $\triangle l$ along the line l will give a change in φ measured by $(\partial\varphi/\partial l)\triangle l$. But the change of $\triangle l$ along l will result in partial changes $\triangle l \cos\alpha$, $\triangle l \cos\beta$, $\triangle l \cos\gamma$ along X, Y, Z. Hence the change $\triangle l \cos\alpha$ in the X direction will give, of itself, a change in φ measured by $(\partial\varphi/\partial x)\triangle l \cos\alpha$ and similarly for the component changes along Y and Z. Hence the sum of these component changes will give the total change, and we shall have:

$$\frac{\partial\varphi}{\partial l}\triangle l = \frac{\partial\varphi}{\partial x}\triangle l \cos\alpha + \frac{\partial\varphi}{\partial y}\triangle l \cos\beta + \frac{\partial\varphi}{\partial z}\triangle l \cos\gamma$$

or

$$\frac{\partial\varphi}{\partial l} = \frac{\partial\varphi}{\partial x}\cos\alpha + \frac{\partial\varphi}{\partial y}\cos\beta + \frac{\partial\varphi}{\partial z}\cos\gamma$$

If, in this equation, we put u, v, and w for the partial derivatives along the axes X, Y, and Z, we shall have:

$$\frac{\partial\varphi}{\partial l} = u\cos\alpha + v\cos\beta + w\cos\gamma \tag{1.3}$$

If then φ is the potential of a vector V with components u, v, w, along X, Y, Z, the expression in (1.3) will be the component of the vector along the line or direction l. Hence more broadly we may say that if φ is a potential for V, then the partial derivative of φ along any direction in space will give the component of the vector along that direction. It is this property of a potential which gives to it its peculiar significance, since in a single algebraic expression for φ there may be thus wrapped up, as it were, a double infinity of vector values—*i. e.*, the vector value *along* any direction *at* any point in space.

For a two-dimensional field, w in (1.3) becomes *zero* and $(\alpha + \beta) = 90^\circ$. Hence, putting θ for the inclination of the vector to X, (1.3) becomes

$$\frac{\partial\varphi}{\partial l} = u\cos\theta + v\sin\theta \tag{1.4}$$

2. Addition Theorem for Velocity Potentials. The characteristic of a function representing a vector potential is that the derivative of the function in any direction shall give the value of the component of the vector in that direction. In particular if φ_1 is such a function and u_1, v_1, w_1 are the three components of the vector in the direction of the axes X, Y, Z, then,

$$u_1 = \frac{\partial \varphi_1}{\partial x}$$

$$v_1 = \frac{\partial \varphi_1}{\partial y}$$

$$w_1 = \frac{\partial \varphi_1}{\partial z}$$

Now suppose that we have a second potential in the same field and let the second function be denoted by φ_2. Then we shall have similar expressions for u_2, v_2, w_2. But u_1 and u_2 are directly additive and also v_1 and v_2, w_1 and w_2 and for the resultant velocity u we shall have:

$$u = u_1 + u_2 = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial x} = \frac{\partial}{\partial x}(\varphi_1 + \varphi_2)$$

and similarly for the other components v and w. Hence it follows that if we have two or any number of functions φ_1, φ_2, φ_3, etc. coexistent in the same field, the resultant function will be the sum of the individual functions and thus,

$$u = \frac{\partial \Sigma \varphi}{\partial x}$$

$$v = \frac{\partial \Sigma \varphi}{\partial y}$$

$$w = \frac{\partial \Sigma \varphi}{\partial z}$$

This result will find important applications in the problems of fluid mechanics.

3. Conditions in Order that a Potential φ may Exist. We must now inquire as to the characteristics which a vector field must present in order that there may exist for it a potential as expressed by the function φ.

We first make the assumption that φ is a potential for the field with vector components u, v, and w. Then,

$$u = \frac{\partial \varphi}{\partial x} \qquad v = \frac{\partial \varphi}{\partial y} \qquad w = \frac{\partial \varphi}{\partial z}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial^2 \varphi}{\partial x \partial y}, & \frac{\partial v}{\partial z} &= \frac{\partial^2 \varphi}{\partial y \partial z}, & \frac{\partial w}{\partial x} &= \frac{\partial^2 \varphi}{\partial z \partial x} \\ \frac{\partial u}{\partial z} &= \frac{\partial^2 \varphi}{\partial x \partial z}, & \frac{\partial v}{\partial x} &= \frac{\partial^2 \varphi}{\partial y \partial x}, & \frac{\partial w}{\partial y} &= \frac{\partial^2 \varphi}{\partial z \partial y} \end{aligned}\right\} \tag{3.1}$$

It is well known in differential calculus (readily verified by trial with any function) that, if φ is a function of x, y, z, then $\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$ and similarly for partial differentiation relative to x and z or y and z. The order makes no difference, the result is the same.

Hence we must have relations as follows:

$$\left.\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0 \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\} \qquad (3.2)$$

Reciprocally, if these conditions respecting u, v, and w are fulfilled, it is clear that there will exist a function φ such that (1.1) will be fulfilled and hence there will exist for such a field distribution of the vector V, a potential φ.

But from Stokes' theorem, (see IX 3) the results of which may be here anticipated, it follows that the conditions of (3.2) are those for a zero value of the line integral of V around an element of area in space at the point x, y, z. And since, furthermore, there are no special limitations on this element of area, and since if the condition holds for any element of area it will hold for any aggregate of such elements we may state the condition of (3.2) as follows:

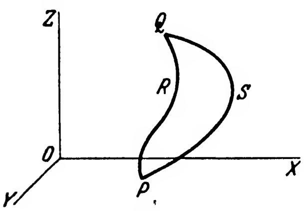

Fig. 41.

If the distribution of the vector is such that its line integral around any closed path in space vanishes, there will exist a potential as expressed by the function φ.

But reference to VI (9.1) will show that the three expressions on the left of (3.2) are the measures of the component rotations of elements of area in the planes of XY, YZ, and ZX about the axes Z, X, and Y respectively. Hence the condition as expressed in (3.2) requires that these component rotations shall all vanish. Hence the rotation of any small element of area in space must vanish, and hence of any small element of volume. Such motion is said to be *irrotational* and a vector field of velocity in which this condition is fulfilled is called an *irrotational* field.

Hence broadly speaking, if the motion is irrotational, there will exist a potential, otherwise not.

The existence of a potential φ as dependent on the vanishing of the line integral of the vector around any closed path in the field may be reached in a somewhat different manner as follows:

In Fig. 41 let P be one point in a space field and Q another. Assume the existence of a potential, which we denote by φ, denote distance along any path between P and Q by s and the vector component determined by taking the derivative along the path, by V. Then,

$$V = \frac{\partial \varphi}{\partial s}$$

or
$$d\varphi = V ds$$

and
$$\varphi\Big]_P^Q = \int_P^Q V ds$$

In words $\triangle\varphi$ is the line integral of V along the path and if starting with a single value of φ at P we are to have a single valu· at Q, it is clear that the value of the integral must be independent of the path followed between the two points. That is, whether the path be PRQ or PSQ, the value of the integral must be the same.

An illustration of this is found in the work done against gravity in carrying a weight W from an altitude H_1 to another H_2. It is clear that the work done against gravity is simply $W(H_2 - H_1)$ and is independent of the path followed in passing from one level to the other.

Referring again to Fig. 41 it follows that the integral along the path PRQ must equal that along PSQ. But the value along QSP must be equal to and opposite in sign to that along PSQ. Hence the integral along PRQ plus the integral along QSP must be zero. But P and Q may be any points in the field. Hence the line integral of V around any closed curve will be zero.

This brings us to the same result as before and we thus conclude that, for a field of fluid motion, if the line integral of the velocity V about any closed curve in the field is zero, the motion is irrotational and a potential will exist. Or otherwise, in order that there may exist a potential relative to the velocity V in a field of fluid motion, the motion must be irrotational.

At this point we must remember that the ideal medium which we are here considering as a fluid is assumed to be incompressible or in any event is assumed to move throughout the field without change of density. For this, VI (6.2) gives the condition

$$\text{Divergence} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{3.3}$$

This is a condition imposed on the vector field quite independent of that for irrotational motion and the question now arises as to the effect of this further condition on the existence and form of a potential function.

But if there is a potential φ, we must have:

$$u = \frac{\partial \varphi}{\partial x} \qquad v = \frac{\partial \varphi}{\partial y} \qquad w = \frac{\partial \varphi}{\partial z}$$

and (3.3) becomes
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{3.4}$$

This is, then, the further condition which any potential function must fulfil.

The expression in (3.4) as an operation or operator is usually designated by the symbol ∇^2 and is known as the *Laplacian* of the vector field. We may therefore express the condition of (3.4) in the symbolic form:
$$\nabla^2\varphi = 0 \text{ or Div. } V = 0 \tag{3.5}$$

The function φ must, therefore, be of such form as to meet the two sets of conditions: (1.1) and hence (3.2) on the one hand and (3.3) or (3.4) on the other.

4. Conditions for the Existence of a Velocity Potential in a Two-Dimensional Vector Field. If we are dealing with a vector field of two dimensions—for example, a velocity V with component velocities along X and Y represented by u and v, the conditions expressed in (3.2) reduce simply to
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.1}$$
and reference to VI (8.2) will again show that this is simply the mathematical expression for irrotational motion in two dimensions, or otherwise, the condition that the line integral of the vector around any closed area in the plane shall vanish.

In the same manner as in **3**, the condition that φ may represent a vector with divergence zero, or in the present case a field of flow as of an incompressible fluid, becomes simply
$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \text{or} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} &= 0 \end{aligned}\right\} \tag{4.2}$$

5. The Functions φ and ψ of Chapter I as Potential Functions for a Two-Dimensional Field. The properties of the functions φ and ψ have been developed in I **2** and we have now to compare these with the properties required for a potential function in a two-dimensional field.

The basic condition for the existence of a potential in a two-dimensional field is expressed by (4.1), which, for a vector representing motion, is simply the mathematical expression of the requirement that the motion shall be irrotational.

But for any function φ, as in Chapter I, we have the relation
$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = 0 \tag{5.1}$$

But this is just the condition of (4.1) and hence it appears that to every function φ there will correspond a vector field with components along X and Y defined by $u = \partial\varphi/\partial x$ and $v = \partial\varphi/\partial y$, and *vice versa*, for every vector field fulfilling the condition of (4.1), there will exist a potential φ, the expression for which may be determined from a knowledge of the field distribution of the components u and v.

But, as we have seen in Chapter I, the two functions φ and ψ cannot occur independently. They are correlative with like properties and the one must always appear with the other; and from the reciprocal relations of these two functions as there set forth, it appears that to every function ψ there will correspond a vector field with components along X and Y defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$; and *vice versa*, corresponding to every vector field fulfilling the condition of (4.1), there will exist a function ψ, related to the components u and v as above, and the expression for which may be determined from a knowledge of the field distribution of the components u and v.

Furthermore for any function φ we have the relation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{(see I 2)}$$

with a similar relation for the function ψ. But this is the condition for incompressible or constant volume flow.

It thus appears in general that either function φ or ψ will meet the requirements for service as a potential for the steady irrotational movement of an incompressible fluid.

Now assume that the necessary field conditions are fulfilled and that φ is a velocity potential with its companion function ψ related as in I 2. Then from either of I (2.7) we have

$$\frac{dw}{dz} = u - i\,v \tag{5.2}$$

This relation will find important applications at a later point.

From I (2.6) we have also the relations:

$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{5.3}$$

whence for the complete differentials,

$$\left.\begin{aligned} d\varphi &= u\,dx + v\,dy \\ d\psi &= u\,dy - v\,dx \end{aligned}\right\} \tag{5.4}$$

In polar coordinates, as in VI 2, we use n and c to denote respectively the value of the vector, or specifically the velocity along and $\perp$ to the radius vector.

Then if there exists for this vector distribution a potential, we must have

$$n = \text{derivative of } \varphi \text{ along } r$$
$$c = \text{derivative of } \varphi \text{ along } \perp \text{ to } r$$

But a small change along r will be represented at the limit by dr and hence the partial derivative will be $\partial\varphi/\partial r$. Similarly a small change $\perp$ to r will be represented at the limit by $r\partial\theta$ and the partial derivative

will be $\partial \varphi / r \partial \theta$. Taking this together with the general relation established in VIII 4 we shall then have, see Fig. 42,

$$\left.\begin{aligned} n &= \frac{\partial \varphi}{\partial r} = \frac{\partial \psi}{r \partial \theta} \\ c &= \frac{\partial \varphi}{r \partial \theta} = -\frac{\partial \psi}{\partial r} \end{aligned}\right\} \tag{5.5}$$

Whence, similar to (5.4)

$$\left.\begin{aligned} d\varphi &= n dr + c r d\theta \\ d\psi &= n r d\theta - c dr \end{aligned}\right\} \tag{5.6}$$

6. Given the Function w, Required to Find the Remaining Functions and the Field. There are two major problems in connection with vector fields; first, given the functions in whole or in part, to determine the field; and second, given the field, to determine its functions.

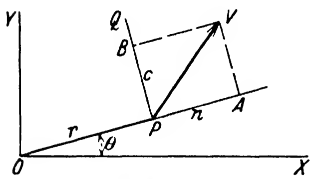

Fig. 42.

First suppose we have given the function w. We write the continued equation

$$w = f(z) = \varphi + i\psi = f_1(x, y) + i f_2(x, y) \tag{6.1}$$

If then we have given $w = f(z)$, we may proceed by expanding $f(z) = f(x + iy)$, separating it into the real and imaginary parts and thus determine φ and ψ and hence u and v by (5.3).

Thus let $\quad w = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$

Then

$$\left.\begin{aligned} \varphi &= x^2 - y^2 \\ \psi &= 2xy \end{aligned}\right\} \tag{6.2}$$

and from (5.3)

$$u = 2x$$
$$v = -2y$$

Or otherwise we find $dw/dz = f'(z) = f'(x + iy)$. We then expand the function, separate the two parts, real and imaginary, equate to $u - iv$, [see (5.2)] and thence find u and v.

Thus in the preceding example,

$$\frac{dw}{dz} = 2z = 2(x + iy) = u - iv$$

Whence

$$u = 2x$$
$$v = -2y$$

the same as the values resulting from (6.2).

7. Given the Function φ or ψ, Required to Find the Remaining Functions and the Field. Suppose φ given. Then from (5.3) we find u and v.

Then from (5.4) we have

$$d\psi = u dy - v dx \tag{7.1}$$

Integrating this as in Chapter II, we find ψ.

In case ψ alone is given we have similarly from (5.4)

$$d\varphi = u\,dx + v\,dy \tag{7.2}$$

which is to be treated in the same manner.

Then uniting φ and ψ in the form

$$\varphi + i\psi = f_1(x, y) + i f_2(x, y)$$

and recasting into the form $f(x + iy)$, we have,

$$w = f(x + iy) = f(z)$$

Or otherwise we put the values of u and v in the form $u - iv$, arrange this in the form $f'(x + iy)$ and then from (5.2) write,

$$\frac{dw}{dz} = f'(x + iy) = f'(z)$$

and thence by integration find w.

Thus for illustration, suppose we have

$$\varphi = (x^2 - y^2)$$

Whence $$u = \frac{\partial \varphi}{\partial x} = 2x$$

$$v = \frac{\partial \varphi}{\partial y} = -2y$$

Then from (7.1) or (5.4)

$$d\psi = u\,dy - v\,dx = 2x\,dy + 2y\,dx$$

whence integrating relative to y we find

$$\psi = 2xy \tag{7.3}$$

The same result is found if we integrate relative to x and hence (7.3) gives the complete integral of $d\psi$ or ψ.

We then write

$$w = \varphi + i\psi = x^2 - y^2 + 2i\,xy = (x + iy)^2$$

Whence $w = (x + iy)^2 = z^2$

Or again we write

$$\frac{dw}{dz} = u - iv = 2(x + iy) = 2z$$

Whence $w = z^2$ as before.

If, on the other hand we had given $\psi = 2xy$ we should find from (5.3)

$$u = 2x$$

$$v = -2y \text{ as before}$$

Then $$d\varphi = 2x\,dx - 2y\,dy$$

which integrates as $\varphi = x^2 - y^2$

thus giving the same functional and field values as before.

By these means, therefore, with either φ or ψ known, the other field functions as well as the velocity distributions may be found.

Let us now represent a particular function φ by A and its mate ψ by B. Then for these special values

$$f(z) = \varphi + i\psi = A + iB$$

Then if A is known, B can be found and *vice versa*. Suppose, however, that we should choose to take A for the ψ function. What then will be the form of the φ function?

From (7.1) where A is taken for the φ function, we have

$$d\psi = -\frac{dA}{dy}dx + \frac{dA}{dx}dy \tag{7.4}$$

And similarly from (7.2) if A is taken for the ψ function:

$$d\varphi = \frac{dA}{dy}dx - \frac{dA}{dx}dy \tag{7.5}$$

We know that the integration of (7.4) will give B; hence the integration of (7.5) will give $-B$.

In the same way it may be shown that if B is taken for the φ function, then $-A$ will result for the ψ function. It is also obvious that if we take $-A$ for the φ function we shall have $-B$ for the ψ function. It thus appears that with the pair of coordinate functions A and B, we may have four combinations as follows:

$$\left.\begin{array}{ll} \text{(a)} & w_1 = f(z) = A + iB \\ \text{(b)} & w_2 = if(z) = -B + iA \\ \text{(c)} & w_3 = -f(z) = -A - iB \\ \text{(d)} & w_4 = -if(z) = B - iA \end{array}\right\} \tag{7.6}$$

It may be noted that these follow in order from the first by succesive multiplications by the factor i.

Using for illustration the same function as above and taking $A = (x^2 - y^2)$, $B = 2xy$, we shall find as in (7.6)

$$\left.\begin{array}{ll} \text{(a)} & w_1 = (x+iy)^2 = (x^2-y^2) + 2ixy \\ \text{(b)} & w_2 = i(x+iy)^2 = -2xy + i(x^2-y^2) \\ \text{(c)} & w_3 = -(x+iy)^2 = -(x^2-y^2) - 2ixy \\ \text{(d)} & w_4 = -i(x+iy)^2 = 2xy - i(x^2-y^2) \end{array}\right\} \tag{7.7}$$

8. Given a Field of Velocity Distribution as Determined by u and v, to Find φ, ψ and w. From (5.3) we have:

$$\left.\begin{array}{l} d\varphi = u\,dx + v\,dy \\ d\psi = -v\,dx + u\,dy \end{array}\right\} \tag{8.1}$$

Then expressing u and v as functions of x and y, integrating udx relative to x and vdy relative to y and rejecting duplicates, we shall have φ. Similarly with $-vdx$ and udy, we find ψ. We then proceed as in **7** to find the form of the function w. Or again we may as in **7** write immediately the equation, $\dfrac{dw}{dz} = u - iv$

and proceed as there indicated to find w and thence φ and ψ.

Thus if we take $u = 2x$ and $v = -2y$ as in 6 and 7, the procedure above indicated will give immediately,

$$\begin{aligned} \varphi &= x^2 - y^2 \\ \psi &= 2xy \\ w &= (x + iy)^2 = z^2 \end{aligned}$$

9. Illustrations of 6, 7, 8. Some further simple illustrations of 6, 7 and 8 may here be given.

(a) Let
$$\begin{aligned} w &= (a + ib)z = (a + ib)(x + iy) \\ \varphi &= (ax - by) \\ \psi &= (bx + ay) \\ u &= a \\ v &= -b \end{aligned}$$

or
$$\frac{dw}{dz} = a + ib = u - iv$$

whence u and v as above.

(b) Let
$$\begin{aligned} w &= \log z = (\log r + i\theta) \quad \text{[see V (4.1)]} \\ \varphi &= \log r \\ \psi &= \theta \\ n &= \frac{1}{r} \\ c &= 0 \end{aligned}$$

whence
$$\begin{aligned} u &= \frac{\cos\theta}{r} \quad \text{[see VI (2.9)]} \\ v &= \frac{\sin\theta}{r} \end{aligned}$$

Or otherwise: $\dfrac{dw}{dz} = \dfrac{1}{z} = \dfrac{1}{x + iy} = u - iv$

Whence as in I 14 (a) we find

$$u - iv = \frac{(x - iy)}{r^2}$$

and
$$\begin{aligned} u &= \frac{x}{r^2} = \frac{\cos\theta}{r} \\ v &= \frac{y}{r^2} = \frac{\sin\theta}{r} \quad \text{as above.} \end{aligned}$$

(c)
$$\begin{aligned} w &= \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{r^2} \\ \varphi &= \frac{x}{r^2} = \frac{\cos\theta}{r} \\ \psi &= -\frac{y}{r^2} = -\frac{\sin\theta}{r} \end{aligned}$$

Whence $n = -\frac{\cos\theta}{r^2}$ [see VII (5.5), VI (2.10)]

$$c = -\frac{\sin\theta}{r^2}$$

and $u = n\cos\theta - c\sin\theta = -\frac{(\cos^2\theta - \sin^2\theta)}{r^2}$

$$v = n\sin\theta + c\cos\theta = -\frac{2\sin\theta\cos\theta}{r^2}$$

or $u = -\frac{(x^2 - y^2)}{r^4}$

$$v = -\frac{2xy}{r^4}$$

Still otherwise (see V 4)

$$\frac{dw}{dz} = -\frac{1}{z^2} = -\frac{1}{r^2 e^{2i\theta}} = -\frac{e^{-2i\theta}}{r^2}$$

or $\frac{dw}{dz} = -\frac{\cos 2\theta - i\sin 2\theta}{r^2} = u - iv$

whence u and v as above.

CHAPTER VIII

POTENTIAL-CONTINUED

1. Interpretation of φ. The basic definition of φ is through an equation of the form:

$\frac{\partial\varphi}{\partial s}$ = component velocity along direction $s = V_s$ where s denotes any line or direction in space. Hence

$$\varphi = \int V_s\, ds$$

But the right hand side is the expression for the line integral of a vector V along a line s. Hence φ as a physical concept, must have the nature of a line integral.

In a more detailed analytical manner, this may be shown as follows:

From VII (5.3),
$$\left.\begin{aligned} u &= \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad (1.1)$$

whence $udx + vdy = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$

But the right hand member is the complete differential of φ, and we may write, therefore,

$$d\varphi = udx + vdy$$

or $\varphi] = \int(udx + vdy) = \int udx + \int vdy$ (1.2)

Suppose now that we have a vector field for which there is a velocity potential φ. Then let Fig. 43 represent the series of curves resulting

from putting $\varphi = \varphi_1$, φ_2, etc., and let APB be any path in this field. Let us then apply (1.2) to this path. But the right hand member is seen to be the line integral of the vector along the path [see VI (4.1)], and hence we reach the result that the difference in the value of φ for the two ends of the path will be a measure of the line integral along the path; or *vice versa*, that the line integral along any path will be the measure of the change in φ between the two ends of the path.

We have thus identified the function φ as one, the change in which, between two points in its vector field, is a measure of the line integral of the vector along a path connecting these two points, as APB in

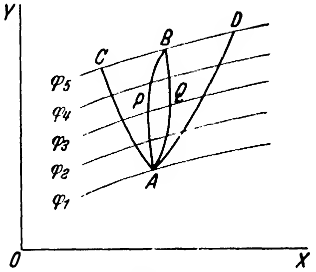

Fig. 43.

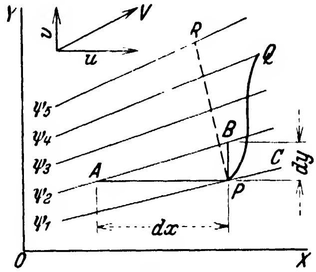

Fig. 44.

Fig. 43. But the change in the value of φ between A and B is the same regardless of the path followed and hence will be the same for APB and AQB. More broadly, the change in the value of φ between φ_1 and φ_5, for example, is the same regardless of the path followed, and hence the same for AD, AQB, APB, or AC.

This is simply another mode of approach for a two-dimensional field to the results of VII 3, see Fig. 41.

2. Interpretation of ψ. We have as the basic relations between ψ and the vector,

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \text{[see VII (5.3)]} \tag{2.1}$$

Suppose now that we have a vector field for which there exists the potential φ and hence the potential function $w = \varphi + i\psi$. Let Fig. 44 denote, in such a field, a series of lines resulting from putting $\psi = \psi_1$, ψ_2, etc., and suppose the values of ψ to increase by regular increments from ψ_1 to ψ_5, etc.

Then for the line $\psi = \psi_1$, let us take the complete differential of ψ:

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

or

$$-v\,dx + u\,dy = 0$$

or

$$\frac{dy}{dx} = \frac{v}{u}$$

Let APB represent on an enlarged scale the conditions at any point in the field. Then along the line AB, this equation will apply, which means that AB lies along the line of the resultant flow V.

This is a result of great importance, showing, as it does, that where the vector represents a field of fluid motion, the direction of stream flow at any point will lie along the line of $\psi =$ *constant* at that point. The lines $\psi =$ *constant* give, therefore, the direction of flow point by point and are therefore called *stream-lines or lines of stream flow*.

We now write again the complete differential of ψ

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

This is the general statement of the make up of a change in ψ due to a small change in location in the field, such change ds having the components dx and dy. From (2.1) this becomes:

$$d\psi = u\,dy - v\,dx \tag{2.2}$$

But in VI 7, this expression was shown to be the flow across the element ds. Hence, integrating along any path from P to Q Fig. 44, we shall obviously have

$$\psi\Big]_P^Q = \text{flow across line } PQ.$$

Or otherwise, $\psi\Big]_P^Q =$ flow between the lines $\psi_1 P$ and $\psi_4 Q$.

Or again $\psi\Big]_P^Q =$ flow between stream-lines passing through P and Q.

If then we take the line $\psi = 0$ as the datum line of the series, the value of ψ for any line $\psi = \psi_m$ will simply equal the total rate of flow between the line $\psi = 0$ and the line $\psi = \psi_m$. Or otherwise, the value of ψ at any point in the field will be measured by the total rate of flow between the line $\psi = 0$ and the line passing through the given point.

It is, however, necessary to adopt some convention regarding the algebraic sign of the flow thus represented by the function ψ. This convention is furnished by the relation between the velocity along the stream-line and the derivative of ψ. In Fig. 44 let $\psi_1 C$ be the datum line from which the flow is to be measured. Let V be the velocity at any point P. Then (see 4 below) V will be measured by the derivative of ψ in the direction $+ 90^0$ with $\psi_1 C$ at P, and hence in order to give V positive in the direction of flow, we must have ψ positive in the direction PR. We thus have the convention that, looking along the datum line in the direction of the flow, positive ψ will lie on the left and conversely, negative ψ will lie on the right.

We have thus established a physical interpretation for the two functions φ and ψ, the one as a line integral and the other as a rate of flow.

Due to its relation to the velocity components, the function φ is often called the *velocity potential*. Similarly the function ψ is commonly

known as the *stream function* while the combination expressed as $w = \varphi + i\psi$ is referred to as the *potential function.*

3. Reciprocal Relations of φ and ψ to a Vector Field. In VII 7 it has been shown that if in a given field the velocity potential is represented by a function A and the stream-line or stream function by a function B, then we may have another vector field in which A is the stream function and $-B$ the velocity potential. It results then that the stream-lines

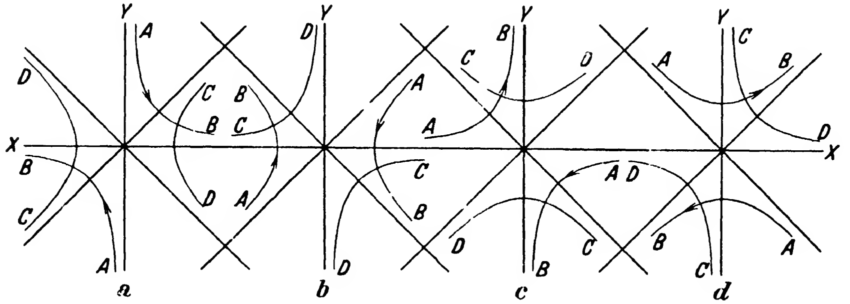

Fig. 45.

in the first field become the lines of constant velocity potential in the second and the lines of velocity potential in the first become the stream-lines in the second.

If we take for illustration

$$f(z) = (x + iy)^2 = (x^2 - y^2) + 2ixy$$

and put $A = (x^2 - y^2)$ and $B = 2xy$, the various equations of VII (7.6) will be represented as in Fig. 45 where a single equipotential and stream-line are shown.

Equation (a) will then be represented by Fig. 45a where AB is the stream-line and CD the equipotential, the direction of flow along AB being as shown by the arrow. Similarly (b), (c), (d) will be represented by Figs. 45 *b*, *c*, *d*. The change from (a) to (b) gives therefore an exchange between stream and equipotential functions and also gives a clockwise rotation of 45° to the axes and similarly in succession for the remaining cases.

4. Geometrical Relation Between Derivatives of the Functions φ and ψ. Writing again VII (5.3), we have

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

The expression $\partial\varphi/\partial x$ implies a partial derivative of φ along the direction of $+X$. The expression $\partial\psi/\partial y$ implies a partial derivative of ψ along $+Y$. But $+Y$ is at an angle of $+90^\circ$ with $+X$. The

first of these two equations states that these two derivatives are equal. In the same manner the expression $\partial\varphi/\partial y$ implies a partial derivative along $+Y$ and $-\partial\psi/\partial x$ one along $-X$. But $-X$ is at an angle of $+90^0$ with $+Y$ and again these two derivatives are equal.

The equations show therefore, that the component u along X may be found by taking the partial derivative of φ along X or that of ψ along a direction at $+90^0$ with X; and similarly that the component along Y may be found by taking the partial derivative of φ along Y or that of ψ along a direction at $+90^0$ with Y.

We have now to show that these relations are general.

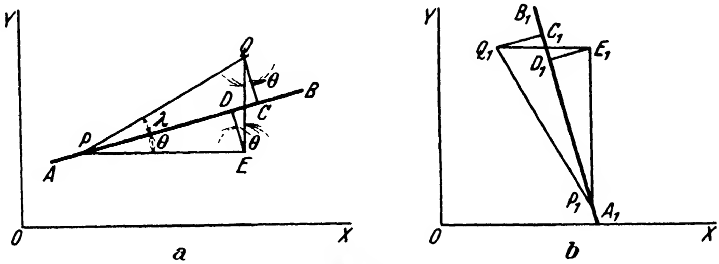

Fig. 46.

In Fig. 46a let PQ be a given vector at the point P and AB any line through P. Then the component of the vector along AB will be:

$$PC = PQ\cos\lambda = PE\cos\theta + EQ\sin\theta$$

or

$$PC = \frac{\partial\varphi}{\partial x}\cos\theta + \frac{\partial\varphi}{\partial y}\sin\theta$$

In this equation, the right hand side is, by definition [see VII (1.4)] the derivative of the function φ along AB.

Next in Fig. 46b, let P_1 be the same point as in a. Then draw A_1B_1 through P_1 and in a direction $\perp$ to the AB of a. Then starting from P_1 suppose we find $\partial\psi/\partial y$. This will equal PE of Fig. 45a numerically, but must be laid off as P_1E_1 in b. We next find $-\partial\psi/dx$. This will equal EQ of Fig. 45a numerically, but must be laid off as E_1Q_1. Then joining P_1Q_1 it is clear that the entire diagram of b is at an angle of $+90^0$ with that of a and that P_1C_1 will equal PC. But P_1C_1 is clearly the partial derivative of ψ along a line A_1B_1, which line is at an angle of $+90^0$ with AB.

It thus results in general that, in a vector field we may, at a given point P, find the component of the vector along any line AB:

(1) By taking the partial derivative of the function φ at P along the line AB.

(2) By taking the partial derivative of the function ψ at P along a line at $+90^0$ with AB.

5. Velocity Relations in an Orthogonal Field of φ and ψ. Let the diagram of Fig. 47 denote a portion of a field laid out with regular series of equipotential and stream-lines or otherwise for a uniformly spaced series of lines for $\varphi =$ *constant* and $\psi =$ *constant*. Then as we have seen, such lines will cross at 90^0 and will form an orthogonal field. Let the direction of flow be as indicated by the arrow. Then at any point P, for example, we shall have the total velocity lying along the line PQ. Furthermore, the products $V_P \cdot AB$ and $V_Q \cdot BC$ denote increments of the line integral of the velocity along the path. But as we have seen in 1 increments of line integral are likewise increments of the function φ. But by assumption these increments are equal. Hence,

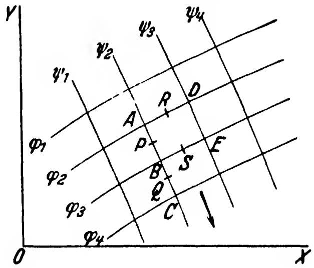

Fig. 47.

$$V_P \cdot AB = V_Q \cdot BC$$

or

$$\frac{V_P}{V_Q} = \frac{BC}{AB}$$

Or in words, in a field laid out as indicated, the velocity along the stream-line from point to point is inversely proportional to the lengths of the intercepts on the line ψ, cut off between successive lines $\varphi =$ *constant*.

Again the products $V_R \cdot AD$ and $V_S \cdot BE$ denote the rate of flow across AD and BE. But by the principle of continuity (incompressible fluids), these must be equal. Hence,

$$V_R \cdot AD = V_S \cdot BE$$

or

$$\frac{V_R}{V_S} = \frac{BE}{AD}$$

Or in words, in a field laid out as indicated, the velocity at any point is inversely proportional to the lengths of the intercepts on successive lines $\varphi =$ *constant* cut off between adjacent lines $\psi =$ *constant*.

In any field with finite subdivision, these relations are obviously only approximately true, increasing in accuracy with the approach of the subdivision to the limit of fineness.

CHAPTER IX

SPECIAL THEOREMS

1. Gauss' Theorem. In Fig. 48 let ABC denote the outline of a closed surface in a three-dimensional vector field referred to axes X, Y, Z. Let the dotted lines represent lines of vector flux through the volume ABC. Let V denote the strength of the vector at any point on the surface ABC. Suppose likewise the surface divided up into small elements dS. Then in general the direction of the vector will

be oblique to the element dS on which it impinges, see Fig. 49. Let θ be the angle made by the vector with the normal to the element. Then $V \cos\theta$ is the component of the vector along the normal N and $V dS \cos\theta$ is a measure of the vector flux through the element dS. Then if flux from without inward across the surface be called positive and from within outward negative, it is clear that a part of the elements of the flux will be positive and a part negative and if a process of integration or summation is assumed, we shall have, as a result, the net flux across the surface.

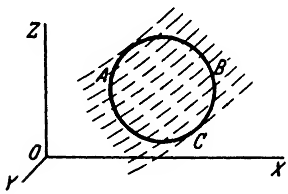

Fig. 48.

Now the purpose of *Gauss' theorem* is to establish a relation between this net flux, as the result of an integration over the surface of the volume, and an integration of the rate of change of the strength of the vector, extended throughout the volume inclosed by the surface.

To make the case general, we suppose the vector to have components P, Q, and R in the directions of X, Y, and Z. We now fix attention on the component P. This is, in itself, a vector parallel to X and for the time being we consider this as the only vector with which we are concerned.

Suppose two sets of planes, Fig. 50, one parallel to XZ and the other parallel to XY and cutting out of the volume an elementary

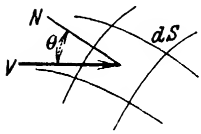

Fig. 49.

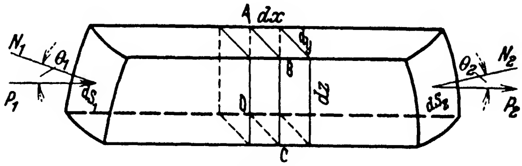

Fig. 50.

parallelepiped with cross section $dy\,dz$ as shown. This element of the volume will be bounded at its two ends by two elements of the surface dS_1 and dS_2. Let N_1 and N_2 denote the normals to these elements, θ_1, θ_2 the angles made with the axis of X, and P_1, P_2 the values of P at dS_1, dS_2.

Then considering only P, the flux through dS_1 and dS_2 will be $P_1 \cos\theta_1\, dS_1$ and $P_2 \cos\theta_2\, dS_2$. Then if we consider the flux to be inward at dS_1 and outward at dS_2 we shall have as the net flux for these two elements of the surface,

$$P_1\, dS_1 \cos\theta_1 - P_2\, dS_2 \cos\theta_2 \tag{1.1}$$

We turn now to the volume integration, and suppose the elementary parallelepiped between dS_1 and dS_2 to be divided into small block elements by planes parallel to YZ, as shown on the diagram.

Then the net flux of the vector P through any one such block element with dimensions dx, dy, dz, will be

$$-\frac{dP}{dx}(dx\,dy\,dz) \text{ (see VI 6).}$$

Hence $\quad -\int \frac{dP}{dx} dx\,(dy\,dz) =$ summation of change of flux for the assemblage of small elements making up the parallelepiped.

This is obviously a volume integration. But for the parallelepiped, $dy\,dz$ is constant and we have for the integral, simply,

$$-dy\,dz \int_{P_1}^{P_2} dP = (P_1 - P_2)\,dy\,dz \tag{1.2}$$

But referring again to Fig. 50

$$dy\,dz = dS_1 \cos\theta_1 = dS_2 \cos\theta_2$$

Hence we may write (1.1) in the form

$$(P_1 - P_2)\,dy\,dz$$

which is the same as (1.2). It thus appears that the summation of the small elements $\frac{\partial P}{\partial x}\,dx\,(dy\,dz)$ or $\frac{\partial P}{\partial x}\,dx\,dy\,dz$ through the volume of the parallelepiped leads to the same result as with the surface expression in (1.1).

But $dx\,dy\,dz$ is a small element of volume. Hence we may express the latter form as the integration of the rate of change of the vector strength throughout the volume under consideration, and thus finally, for the elementary parallelepiped, we may write:

$$\begin{aligned} -\int \frac{\partial P}{\partial x}(dx\,dy\,dz) &= -\int \frac{\partial P}{\partial x} dx\,(dy\,dz) \\ &= (P_1 - P_2)\,dy\,dz \\ &= P_1\,dS_1 \cos\theta_1 - P_2\,dS_2 \cos\theta_2 \end{aligned} \tag{1.3}$$

A somewhat different geometrical picture of this volume integration may be gained as follows: The expression

$$\frac{\partial P}{\partial x} dx\,(dy\,dz)$$

is the difference between the vector flux *in* through one face $dy\,dz$ of the small element, and the flux *out* through the other face. For the component vector P, there is, of course, no flux through the faces of the element which are parallel to X.

This expression is therefore a measure of the net flux through the surface of the element $dx\,dy\,dz$. Consequently the summation

$$-\int \frac{\partial P}{\partial x}\,dx\,(dy\,dz) \quad \text{or} \quad -\int \frac{\partial P}{\partial x}(dx\,dy\,dz)$$

is, in effect, the summation of all these elementary net flows for the elementary parallelepiped as a whole. From this viewpoint, (1.3) asserts

that the net vector flux through the boundary ends of a parallelepiped as in Fig. 50 is the same as the net sum of the elementary vector fluxes through the surfaces of the small volume elements into which the parallelepiped may be assumed to be divided.

It is clear, however, that the vector flux through any face $ABCD$ is in effect the flux *outward* from the element lying on its left and *inward* to the element lying on its right. In the summation for both elements therefore it will appear once with the (+) sign and once with the (—) sign and the two values will cancel. Hence the flux across all such inner surfaces (those forming a boundary between two elements) will vanish in the integration and there will remain only the flux inward across the end face at the left and the flux outward across the end face at the right. And that, as we have seen, will be expressed by $+ P_1\, dy\, dz$ and $- P_2\, dy\, dz$ and the sum by $(P_1 - P_2)\, dy\, dz$. Compare VI 4, Fig. 25.

Thus far the proof has considered only a single element of volume and a single vector P parallel to the axis of X. Entirely similiar results will hold for the collection of such elements and likewise for the component vectors Q and R, parallel to Y and Z. In its most general form the relation may therefore be written:

$$\left.\begin{aligned} &\iint (P \cos \alpha + Q \cos \beta + R \cos \gamma)\, dS = \\ &= -\iiint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz \end{aligned}\right| \qquad (1.4)$$

the first integration being extended over the surface and the second throughout the volume.

If P, Q, and R denote the component velocities in a field of fluid motion, the parenthesis in the left hand member of the equation represents the three component velocities resolved along the normal to the surface— that is, the total component velocity along the normal. But, assuming the field to be irrotational and with a velocity potential φ, this total component will be $\partial\varphi/\partial n$ where n denotes direction inward along the normal.

Again on the right the expression within the parenthesis becomes the divergence of the field represented, as in VII 3, by $\nabla^2\varphi$. With this understanding, therefore (1.4) becomes

$$\iint \frac{\partial \varphi}{\partial n}\, dS = -\iiint \nabla^2 \varphi\, dx\, dy\, dz \qquad (1.5)$$

This is the statement in mathematical terms that the gain or loss within a closed boundary in a field of vector flux is equal to the net flow across such boundary. If the vector represents the velocity in a field of fluid motion, the gain or loss of fluid within the boundary is equal to the net flow across the boundary. If the fluid is incompressible, we have $\nabla^2 \varphi = 0$ and hence no gain or loss and hence no net flow across the boundary

2. Green's Theorem. In the preceding section, P, Q, and R were assumed to be components of a vector field and hence vectors in themselves. The proof of (1.4) is, however, quite independent of the particular nature of these components. It is only necessary that they should be finite, continuous and single valued at all points in the region under consideration.

Let us then assume two functions φ_1 and φ_2, each, with their first derivatives, having these same characteristics. We may then put respectively,

$$P, Q, R = \varphi_1 \frac{\partial \varphi_2}{\partial x}, \varphi_1 \frac{\partial \varphi_2}{\partial y}, \varphi_1 \frac{\partial \varphi_2}{\partial z}$$

Then with $\cos\alpha$, $\cos\beta$, $\cos\gamma$, as the direction cosines for $\partial\varphi_2/\partial x$, $\partial\varphi_2/\partial y$, $\partial\varphi_2/\partial z$, with the normal, the left hand side of (1.4) becomes the integral of $\varphi_1 \frac{\partial\varphi_2}{\partial n}$ over the boundary surface, where, as before, n denotes direction along the normal.

Likewise differentiating the values of P, Q, and R, respectively with regard to x, y, and z, we shall have finally

$$\iint \varphi_1 \frac{\partial \varphi_2}{\partial n}\, dS = -\iiint \left(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y}\frac{\partial \varphi_2}{\partial y} + \frac{\partial \varphi_1}{\partial z}\frac{\partial \varphi_2}{\partial z}\right) dx\, dy\, dz$$
$$- \iiint \varphi_1 \nabla^2 \varphi_2\, dx\, dy\, dz \qquad (2.1)$$

Evidently φ_1 and φ_2 may be interchanged giving the same equation as in (2.1), but with interchange of subscripts. If, however, φ_1 and φ_2 represent, as velocity potentials, two modes of flow of an incompressible fluid, we shall have $\nabla^2 \varphi_1$ and $\nabla^2 \varphi_2$ each equal to zero and the interchange of subscripts in the two equations will give

$$\iint \varphi_1 \frac{\partial \varphi_2}{\partial n}\, dS = \iint \varphi_2 \frac{\partial \varphi_1}{\partial n}\, dS$$

Again making $\varphi_1 = \varphi_2 = \varphi$ and assuming $\nabla^2\varphi = 0$, (2.1) becomes

$$\iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz = -\iint \varphi \frac{\partial \varphi}{\partial n}\, dS$$

But the left hand side multiplied by $\varrho/2$ and integrated throughout the volume under consideration, is seen to be the kinetic energy of the field. Denoting this by E, we have

$$E = -\frac{\varrho}{2} \iint \varphi \frac{\partial \varphi}{\partial n}\, dS \qquad (2.2)$$

This is a result of great importance, giving, as it does, the kinetic energy of the field in terms of a function of the velocity potential, summed over the bounding surface. According to this equation, we have then to find, at each point on the bounding surface, the value of φ and of the velocity inward along the normal, multiply them together and into an element dS of the boundary area, and sum over the surface.

In case the body has a uniform cross section with relative motion at right angles to its length, the problem becomes that of integrating $\varphi\partial\varphi/\partial n$ around the cross section and dS becomes simply the product

hds where h is the length of the body considered. Per unit length of body, therefore, dS is represented simply by the element of length ds around the cross section. We obtain a picture of this case by considering an indefinitely long cylinder (not necessarily of circular cross section) moving at right angles to its length. Remembering the relations between the partial derivatives of φ and ψ in two dimensional motion, we have then, for this case

$$E = -\frac{\varrho}{2}\int \varphi \frac{\partial \varphi}{\partial n}\, ds = -\frac{\varrho}{2}\int \varphi \frac{\partial \psi}{\partial s}\, ds = -\frac{\varrho}{2}\int \varphi\, d\psi \qquad (2.3)$$

where E is the energy for unit length of body.

In the last form $d\psi$ may be taken relative to any variable in terms of which it can be conveniently expressed, and then integrated around the contour of the cross section.

3. Stokes' Theorem. In *Gauss' theorem,* a vector flux over the surface of a volume is expressed in terms of an integration extended throughout the volume inclosed by the surface.

In *Stokes' theorem,* an analogous relation is developed between the line integral of a vector around any closed curve lying on a surface, and a certain surface integral extended over the surface inclosed by such curve.

Let V denote any space vector with components P, Q, R, parallel to the axes X, Y, Z.

Then in analytical form, the statement of Stokes' theorem is as follows:

$$\left.\begin{aligned} &\int(P\,dx + Q\,dy + R\,dz) = \\ &= \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \iint\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \\ &\quad + \iint\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx \end{aligned}\right\} \qquad (3.1)$$

The first integral is seen to be the line integral in the form of VI (3.2) with the integration carried around the closed curve. It will be noted in particular that, referring to any particular element of length ds, the components P, Q, and R are X, Y, and Z components of the vector V *at* that particular point, while dx, dy, dz, are the corresponding components of the length ds.

On the right, the expressions under the sign of integration will be recognized as the line integrals about elements of area lying in planes parallel respectively to YZ, ZX, and XY—see VI (8.1). These integrals are to be extended over the surface in space bounded by the given closed curve. In particular it should be noted that the various derivatives $\partial R/\partial y$ etc. are to be taken at a point *on* the surface and not at points on the planes of projection, YZ, etc.

We have now to show the equivalence of the expressions on the two sides of this equation.

In Fig. 51, let $EFGH$ represent a closed curve drawn on a surface which is cut by the three coordinate planes in curves JK, KL, and LJ. On the surface and within this curve, $ABCD$ is an element of area cut out by two pairs of planes parallel to XZ and YZ. Now at the limit with the element very small, the surface cut out will be sensibly a plane. In Fig. 52, let $ABCD$ be such a plane element of the surface with the axes moved up so that the plane XY touches at A, the plane XZ contains CD and the plane YZ contains BC, while the lines AB and DA lie in planes parallel respectively to XZ and YZ. Such an element of surface will be a

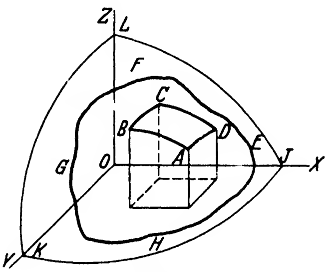

Fig. 51.

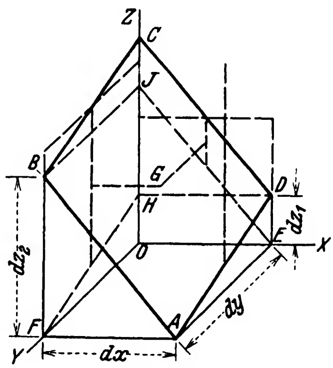

Fig. 52.

parallelogram and its three projections on the coordinate planes will be: a rectangle $EAFO$ on XY, a parallelogram $FBCH$ on YZ and a parallelogram $EJCD$ on XZ. Let G be the midpoint of the element $ABCD$ and let V be the value of the vector at G, with P, Q, and R its components along the directions of X, Y, and Z.

Put $ED = dz_1$ and $FB = dz_2$. Then the area of the parallelogram $EJCD$ is $dx\,dz_1$ and that of the parallelogram $FBCH$ is $dy\,dz_2$. Likewise the point G will project parallel to Y into the center of $EJCD$ and parallel to X into the center of $FBCH$. Then the coordinate steps which must be traversed from G to the midpoint of the four sides of $ABCD$ will be as follows:

$$G \text{ to midpoint of } AB\text{:}\quad \frac{dy}{2} - \frac{dz_1}{2}$$

$$G \text{ to midpoint of } BC\text{:}\quad -\frac{dx}{2} + \frac{dz_2}{2}$$

$$G \text{ to midpoint of } CD\text{:}\quad -\frac{dy}{2} + \frac{dz_1}{2}$$

$$G \text{ to midpoint of } DA\text{:}\quad \frac{dx}{2} - \frac{dz_2}{2}$$

Then remembering the relations developed in VI 4 we may write the elements of line integral around $ABCD$ as follows:

$$P_{AB} = \left[P + \frac{\partial P}{\partial y}\frac{dy}{2} - \frac{\partial P}{\partial z}\frac{dz_1}{2}\right](-dx)$$

$$R_{AB} = \left[R + \frac{\partial R}{\partial y}\frac{dy}{2} - \frac{\partial R}{\partial z}\frac{dz_1}{2}\right](+dz_2)$$

$$Q_{BC} = \left[Q - \frac{\partial Q}{\partial x}\frac{dx}{2} + \frac{\partial Q}{\partial z}\frac{dz_2}{2}\right](-dy)$$

$$R_{BC} = \left[R - \frac{\partial R}{\partial x}\frac{dx}{2} + \frac{\partial R}{\partial z}\frac{dz_2}{2}\right](+dz_1)$$

$$P_{CD} = \left[P - \frac{\partial P}{\partial y}\frac{dy}{2} + \frac{\partial P}{\partial z}\frac{dz_1}{2}\right](+dx)$$

$$R_{CD} = \left[R - \frac{\partial R}{\partial y}\frac{dy}{2} + \frac{\partial R}{\partial z}\frac{dz_1}{2}\right](-dz_2)$$

$$Q_{DA} = \left[Q + \frac{\partial Q}{\partial x}\frac{dx}{2} - \frac{\partial Q}{\partial z}\frac{dz_2}{2}\right](+dy)$$

$$R_{DA} = \left[R + \frac{\partial R}{\partial x}\frac{dx}{2} - \frac{\partial R}{\partial z}\frac{dz_2}{2}\right](-dz_1)$$

In forming these terms care must be taken with regard to the cyclical order around $ABCD$ in relation to the direction of the components P, Q and R along its sides. We have here chosen the cyclical order of rotation from axis to axis of $XYZX$ or $ABCDA$ around the element of area. Then the component of a $+P$ will lie in the direction BA or the reverse of AB. Hence in P_{AB}, dx is given the $(-)$ sign. On the other hand, a $+R$ will have a component along the direction AB and in R_{AB}, dz_2 is given the $(+)$ sign. In this manner the signs for the various terms are determined.

Carrying out the algebraic work and reducing we find for the aggregate of these terms:

$$\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) dy\, dz_2 + \left(\frac{dP}{dz} - \frac{dR}{dx}\right) dz_1\, dx + \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy$$

In these terms, as we have seen, $dx\, dy$, $dz_1\, dx$, $dz_2\, dy$ are the projections of the element of area $ABCD$ on the planes XY, XZ, and YZ respectively.

In summing these expressions over the entire surface, each expression would be taken separately and double integrated, between the proper limits of the variables involved. In such case, there is no longer any significance in retaining the subscripts for dz and the integration of these expressions over the surface may be written as in (3.1).

But the sum of $P_{AB} + R_{AB} + Q_{BC} + \ldots\ldots$ will obviously give the line integral about the element $ABCD$ and for the aggregate of such elements, the values for the internal boundaries will all vanish the same as for a plane area (see VI 4). Hence the sum of the line

integrals for the individual elements such as $ABCD$, Fig. 51 will give the line integral about the outer boundary $EFGH$.

But this is represented by the left hand side of (3.1). We have thus shown that the summation of all the elements which go to make up the line integral as expressed on the left hand side of (3.1), may also be expressed as the series of double integrals on the right of the same equation and thus the proof of the theorem is established.

The establishment of the equality of the two sides of (3.1) enables us further to express the conditions for the vanishing of the line integral around any closed curve in space as follows:

$$\left.\begin{aligned}\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 0\\ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= 0\\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= 0\end{aligned}\right\} \tag{3.2}$$

CHAPTER X

CONFORMAL TRANSFORMATION

1. Introductory. The subject of conformal transformation comprises the study of methods whereby a geometrical field 1, characterized by

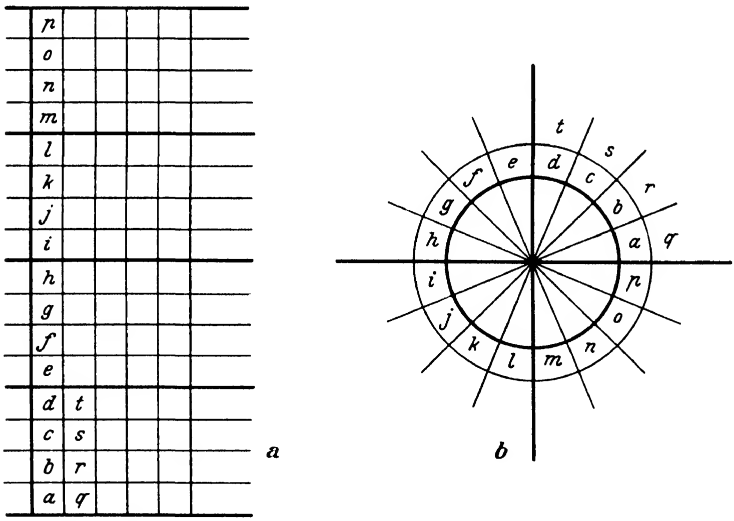

Fig. 53.

an assemblage of points and lines may be transformed into another field 2, point by point and line by line, in such manner that an indefinitely small element of area in field 1 shall transform in field 2 into an element of similar geometrical form and proportions, while at the

same time the aggregate in field 2 may be quite different from that in field 1.

The meaning of this will be made clear by reference to Figs. 53 and 54.

In Fig. 53a, the element is a small rectangle. In Fig. 53b, the element is a quadrilateral figure. If the subdivision should be carried

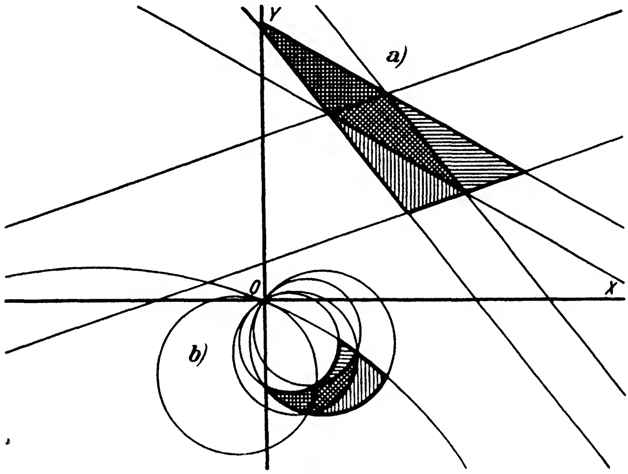

Fig. 54.

to the limit, corresponding elements would be rectangular in each diagram and similar each to each. Nevertheless, the aggregate in the one case is quite different from that in the other. It is thus seen that assemblage *b* may be realized by a transformation of *a* or assemblage *a* by a transformation of *b*, each small element retaining its form and proportion,

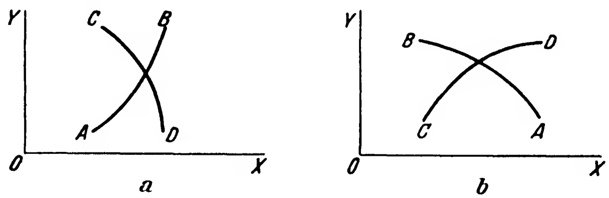

Fig. 55.

but suffering such change in size and disposition as the character of the transformation may require.

In similar manner, Fig. 54, a and b shows the transformation of one field into another where the small elements are triangular in form.

It is seen that if the small elements of area are to remain similar in the transformation, then the angles between any two intersecting lines in field 1 (AB and CD Fig. 55a) will remain the same in the transformed field 2 (AB and CD Fig. 55b).

We have now to determine the character of transformation which will meet these conditions.

2. Application of Vectors to the Problem of Conformal Transformation. From the properties of vectors, as developed in Chapter V, it is clear that a vector may be employed to locate or specify a point in a plane relative to an origin and a pair of rectangular axes. Thus, for example,

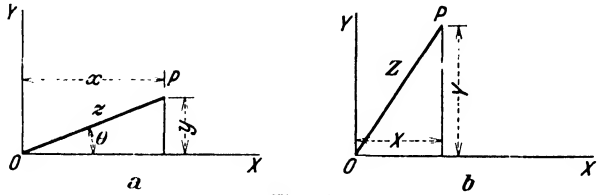

Fig. 56.

the point P in Fig. 56a. In developing the theory of conformal transformation we shall designate a vector in field 1 (the initial or basic field) by the symbol z with components x and y. Thus,

$$z = x + iy$$

denotes the vector for any point P in field 1. Now let us designate a vector in field 2 by Z with components X and Y. Then in field 2 we shall have

$$Z = X + iY$$

as the designation or description of the transformed vector in terms of its components X and Y (see Fig. 56b).

Let us now assume a relationship between the two vectors for corresponding points in the two fields of the form

$$Z = f(z) \tag{2.1}$$

That is, vector Z is a particular and specified function of vector z.

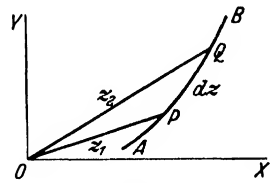

Fig. 57.

It is further to be assumed that $f(z)$ may be any function of z, real, continuous and admitting of expression in algebraic form.

If then we should take a series of points in field 1, with their vectors z, and subject each such vector to the transformation indicated by $f(z)$, we shall have a new series of vectors Z, and if these be then laid off in field 2, we shall have the transformation of the assemblage of points in field 1 into its corresponding assemblage in field 2. In particular if the assemblage in field 1 constitutes some particular line in that field, then will the corresponding assemblage in field 2 represent the corresponding line in the transformed field.

We have now to develop the consequences of this form of relation between the vectors Z and z.

Differentiating (2.1), we have

$$d\,Z = f'\,(z)\,d\,z \tag{2.2}$$

In Fig. 57, $A\,B$ is some given line in field 1. Let P and Q be two points near together on this line and z_1, z_2 the two vectors to these points. Then in a vector sense:

$$z_2 - z_1 = PQ$$

Or at the limit, $d\,z = PQ$, a small element of the line $A\,B$. The point to be noted here is that $d\,z$ in the vector sense is not the difference in length between the two vectors z_1 and z_2 but rather the element of the line $A\,B$, to the ends of which the two vectors are drawn.

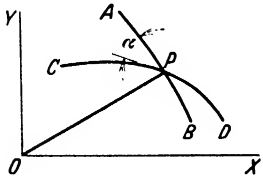

Fig. 58.

Similarly $d\,Z$ will be the corresponding element in the transformed field.

Hence $d\,z$ and $d\,Z$ are here to be viewed simply as corresponding elements of the line $A\,B$, the first in field 1 and the second in field 2.

Now (2.2) gives the relation between these two elements. It shows that $d\,Z$ is to be produced by multiplying $d\,z$ by $f'\,(z)$. But $f'\,(z)$ is a vector and $d\,z$ is a vector. Hence the multiplication of $d\,z$ by $f'\,(z)$ will result in turning $d\,z$ through the angle corresponding to the vector $f'\,(z)$ and then in multiplying its length by the scalar value of $f'\,(z)$.

Using the exponential form of vector expression we may put

$$f'\,(z) = r\,e^{i\theta}$$

where θ is the angle of the vector and r is the scalar value. Hence in the present case (2.2) tells us that the element in field 2 will be given by turning the element $d\,z$ in field 1 through the angle $+\,\theta$ and then multiplying by r. The actual location of the element is not given by this formula—only its direction and length. The location will be given, of course, by the transformed positions of P and Q, but with these we are not now concerned.

Next in Fig. 58 let $A\,B$ and $C\,D$ be any two lines of field 1 intersecting at P, and consider the two indefinitely small elements of these lines lying at this intersection. Suppose both lines subjected to the same transformation, $\quad Z = f\,(z)$

Then the element of $A\,B$, in the transformation, will be rotated through the angle θ and multiplied in length by r, both as determined by $f'\,(z)$. But at the limit, $f'\,(z)$ for the element of $C\,D$ will be the same as that for the element of $A\,B$. Hence the element of $C\,D$ will also be rotated through the same angle θ and multiplied in length by the same multiplier r. We are here only concerned, at the moment, with the fact that both elements will be rotated through the same angle θ, and, of course, located at the transformed point P. Hence in the

transformed field, the angle between these elements will remain unchanged and if such angle be α in field 1, it will remain α in field 2.

Again in Fig. 59 let $ABCD$ denote a small element of area of field 1. Then at the limit, this element will become a small parallelogram located by the vector z. In the transformation, as we have seen, all the angles of this parallelogram will remain the same. Likewise the lengths of sides in field 2 will result from those in field 1 by multiplying by r, the scalar length of $f'(z)$. The point here is that at the limit the factor $f'(z)$ will be the multiplier for all of the elements of the small area $ABCD$ and hence all such elements will be affected in the same ratio as to length. Hence, at the limit, with angles the same and sides in proportion, any such transformed area in field 2 will be geometrically similar to the original area in field 1. This result is readily generalized to include elementary areas of any number of sides or form.

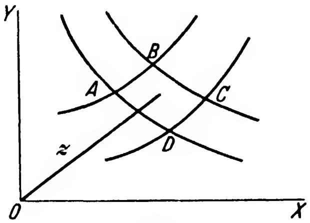

Fig. 59.

It appears therefore that the condition requisite for the realization of "conformal transformation" in the sense as defined in 1 is that the transformation shall be effected through a vector relation of the form $$Z = f(z)$$

Certain special conditions and limitations will appear at a later point.

First, however, some illustrative forms of the transforming functions may be noted

3. Typical Forms which the Transforming Function May Take.

$$Z = z + a \tag{a}$$

$$Z = z + ib \tag{b}$$

$$Z = z + (a + ib) \tag{c}$$

$$Z = m\,z \text{ (where } m \text{ is a number)} \tag{d}$$

$$Z = (a + ib)\,z \tag{e}$$

$$Z = \frac{A}{z} \text{ (where } A \text{ is a number or a vector)} \tag{f}$$

$$Z = A\,z^2 \text{ (where } A \text{ is a number or a vector)} \tag{g}$$

or in general

$$Z = A\,z^n \text{ (where } A \text{ is a number or a vector and } n \text{ is whole or fractional, positive or negative.)} \tag{h}$$

or any combinations of such forms as

$$Z = A + B\,z + C\,z^2 + D\,z^3 + \dots \tag{i}$$

$$Z = A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \dots \tag{j}$$

where A, B, C, etc. may be either numbers or vectors, or again a combination of such forms as in (i) and (j).

The results with some of the simpler of these forms of transformation will now be examined.

4. Illustrative Transformations. It is clear that transformation (a) will result simply in moving the field a distance a in the direction of $+X$, leaving it otherwise the same. Similarly (b) will move the field a distance b in the direction of $+Y$ while (c) will combine these two movements, resulting in the displacement of the field a distance $r = \sqrt{a^2 + b^2}$ in the direction $tan^{-1}\, b/a$. All of these transformations, therefore, leave the field itself unchanged and need not be further considered.

With (d) it is clear that everything will be multiplied by the number m. This will result in the same field, only larger or smaller in dimension in the ratio m, but otherwise unchanged.

With (e), each vector z in the transformation will be turned through the angle $tan^{-1}\, b/a$ and changed in length by the multiplier $r = \sqrt{a^2 + b^2}$,

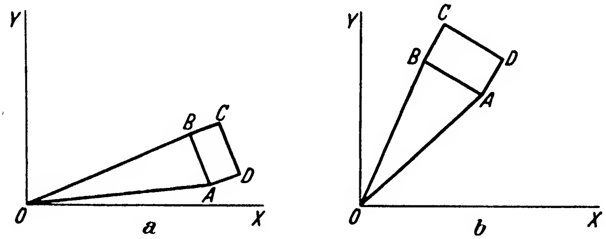

Fig. 60.

thus producing the vector Z. This will then result in turning the entire field through a certain angle and in changing its dimensions in a certain ratio. See Fig. 60.

It will be seen that these various transformations do not change the fundamental character of the field itself. Whatever the geometrical character of the lines making up field 1, those making up field 2 will be the same and the only change will be displacement, a change of scale or a rotation, or some combination of these.

With the remaining forms, these simple relations no longer hold.

With (f) we have the relation that one vector is the reciprocal of the other. Writing z in the exponential form, we have

$$\left.\begin{aligned} z &= re^{i\theta} \\ Z &= \frac{1}{z} = \frac{1}{r}e^{-i\theta} \\ |Z| &= \frac{1}{|z|} = \frac{1}{r} \end{aligned}\right\} \qquad (4.1)$$

$$\theta \text{ for } Z = -\,\theta \text{ for } z$$

or again: $$Z = \frac{1}{z} = \frac{x - i\,y}{x^2 + y^2} = X + i\,Y$$

and $$\left.\begin{aligned} X &= \frac{x}{x^2 + y^2} = \frac{x}{r^2} \\ Y &= \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} \end{aligned}\right\} \tag{4.2}$$

If then we should propose to carry out this transformation by means of geometrical methods, we might proceed according to either mode of expression. Thus we might measure the length z in field 1, take its reciprocal and lay it off along a line in field 2 making an angle of $-\theta$ with the axis of X, thus giving the transformed point P_1. See Fig. 61, a and b. Or again we might measure x and y in field 1 and measure or compute r and then lay off the coordinates in field 2 according to (4.2), thus giving the same point P_1.

5. Transformation of a Field of Lines. It will be desirable at this point to examine the general character of procedure required in order to transform a field of lines from one plane to the other through the use of some given equation of transformation.

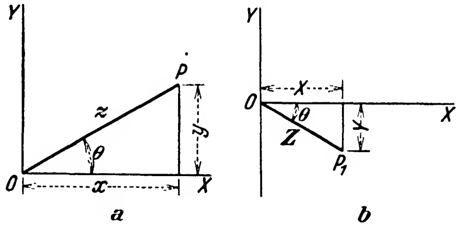

Fig. 61.

This general problem will obviously reduce to the detail of transforming any given line from one plane to the other, and this again to the problem of transforming any given point from one to the other. The latter operation is given by the equation of transformation, $Z = f(z)$.

To realize this end, several methods are available as follows.

(1) By means of a graphical construction based on the vector z, and into the details of which we need not enter, we can construct graphically the vector $f(z)$ and hence Z. See above with reference to (4.1).

(2) We may expand $f(z)$ into the form $Z = f(z) = X + iY$ and thus find X and Y in terms of x and y. See (4.2) for example. We may then, for any point on the z plane compute the coordinates X, Y, on the Z plane and thus determine the transformed point.

(3) By analytical methods, suppose that we have in the plane z, a given line, the equation to which, in x and y, is

$$f(x, y) = A \tag{5.1}$$

where A is a constant. The expansion of the equation $Z = f(z) = X + iY$ will give

$$X = f_2(x, y) \tag{5.2}$$

$$Y = f_3(x, y) \tag{5.3}$$

That is, for any point in general on the plane z, with coordinates x and y, the coordinates of the transformed point on the plane Z, will be X and Y, given as functions of x and y as in (5.2) and (5.3).

It is clear then that the equation to the transformed line on the Z plane will be an equation in X and Y, representing the collection of all points meeting, in plane z, the condition of (5.1) and transformed to plane Z through the coordinate relations given by (5.2) and (5.3). Such an equation between X and Y can obviously be found by eliminating x and y between the three equations (5.1), (5.2), and (5.3).

This will give, for the transformed line, an equation in X and Y of the form:

$$F(X, Y) = A \tag{5.4}$$

Referring now to (5.1), the value of A in any particular case in plane z will identify a particular line in that plane, and this particular line transformed into the Z plane will naturally be identified by the same numerical value of A. In particular will this be the case if to the line in the z plane there attaches some special physical significance or characteristic. We therefore employ the same constant in (5.1) and (5.4) and with the understanding that for corresponding lines in the two planes the value of the constant will be the same. This point is of importance in connection with the application of conformal transformation to problems in fluid mechanics.

Otherwise we may deduce the transformed equation in terms of polar coordinates as follows:

We have:

$$\left.\begin{aligned} Z &= f(z) \\ z &= re^{i\theta} \\ r &= f_1(\theta) \end{aligned}\right| \tag{5.5}$$

The first of these is the equation of transformation. The second is the general expression of a vector in its exponential form. The third is the polar equation to the line to be transformed.

We have then to eliminate z and r between these equations giving a result in the form $\quad Z = F(\theta)e^{i\theta}$

This means a length $F(\theta)$ laid off at an angle θ with X. Hence the usual form of polar equation will be:

$$r = F(\theta) \tag{5.6}$$

where r is the radius vector and θ the angle as in the usual convention for polar coordinates.

6. Illustrative Field Transformations. Take again the equation

$$Z = \frac{1}{z}$$

and let the given field consist of straight lines parallel to X and having the equation $\quad y = k$

Whe shall then have as our three equations

$$X = \frac{x}{x^2 + y^2} \tag{6.1}$$

$$Y = \frac{-y}{x^2 + y^2} \tag{6.2}$$

$$y = k \tag{6.3}$$

To eliminate x and y proceed as follows:

Solve (6.2) for x.

Put the resulting value in (6.1).

Put $y = k$ in this transformed equation.

There will result the equation

$$X = \frac{\sqrt{-k/Y - k^2}}{-k/Y}$$

This readily reduces to $\quad X^2 + Y^2 + \frac{Y}{k} = 0$

And this, by adding $1/4k^2$ to both sides becomes

$$X^2 + \left(Y + \frac{1}{2k}\right)^2 = \left(\frac{1}{2k}\right)^2$$

This will be recognized as the equation to a circle with radius $1/2k$ and with center on the axis of Y and at a distance of $1/2k$ below the origin. Putting, for convenience, both fields on the same axes we have then the circle as shown in Fig. 62 as the transformation of the straight line $A\,B$.

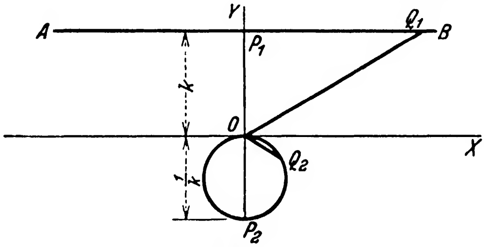

Fig. 62.

The reader will find it of interest to follow out a few selected points by way of geometrical verification. Thus the point P_1 transforms into the point P_2, the point Q_1 into Q_2 etc.

Obviously the same general result will hold for a line parallel to the aixs of Y and by the same method of procedure it may be proven that any straight line $y = mx + b$ will, through the transformation $Z = 1/z$ transform into a circle, and further, that the characteristics of this circle will be as follows:

It will pass through the origin.

Its tangent at the origin will be inclined to X at an angle $-\alpha$, where α is the inclination of the line itself.

Its diameter will lie at right angles to this tangent and will have a length equal to the reciprocal of the perpendicular let fall from the origin to the given line. See Fig. 63.

The algebraic details of this may be left to the interested reader.

It follows, likewise, and this is of importance, that any circle passing through the origin will transform through the reciprocal relation, into a straight line, related to the circle in manner inverse to the relation between the circle and straight line as noted above.

These relations are more readily developed from the exponential vector form than from the component form as above. Thus in Fig. 63

let P be any point on the line. Then, h being the perpendicular distance from the origin to the line, we shall have as the polar equation to the line:

$$r = h/\sin(\theta - \alpha) = h \operatorname{cosec}(\theta - \alpha) \tag{6.4}$$

Equations (5.5) then become $Z = \dfrac{1}{z}$

$$z = re^{i\theta}$$

$$r = h \operatorname{cosec}(\theta - \alpha)$$

And from these we derive immediately,

$$Z = \frac{\sin(\theta - \alpha)}{h} e^{-i\theta}$$

But $(1/h) \sin(\theta - \alpha)$ is readily seen to be the radius vector of a circle of diameter $1/h$, and remembering that $e^{-i\theta} = \cos\theta - i \sin\theta$, it follows that as a vector, Z must be laid off at the angle $-\theta$. This will give a circle as in Fig. 63. When $\alpha = 0$ we have the case of a line parallel to x and the conditions of Fig. 62 result.

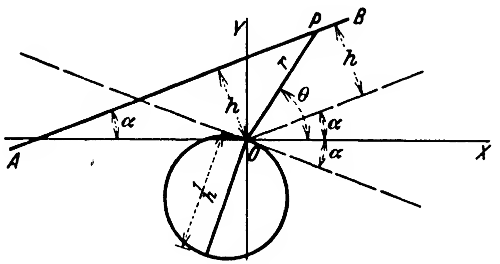

Fig. 63.

A field of lines as in Figs. 62 and 63 will therefore transform into a field of circles determined as shown, or again a field of such circles will transform back into a field of parallel lines.

Take next the relation $Z = z^2$

and let the field consist of lines having the equation

$$y = mx + b$$

Taking again the polar form, we have for the line, as in the last case,

$$r = h \operatorname{cosec}(\theta - \alpha)$$

Then $$Z = z^2 = r^2 e^{2i\theta} = r^2 e^{i2\theta}$$

or $$Z = h^2 \operatorname{cosec}^2(\theta - \alpha)\, e^{i2\theta} \tag{6.5}$$

Now (6.5) means that as a vector Z will have a length $h^2 \operatorname{cosec}^2(\theta - \alpha)$ laid off at an angle 2θ. The normal form of the equation to a line in polar coordinates is of the form

$$r = f(\theta)$$

where the distance $f(\theta)$ is to be laid off at the angle θ. Here, however, we have the distance expressed in terms of θ while the angle at which it is to be laid off is 2θ. Equation (6.5) cannot, therefore, be interpreted directly as an equation in polar coordinates. It will, however, admit

of such interpretation by expressing it all in terms of one angle—the angle 2θ at which the length of the vector is to be laid off.

Remembering that $2 \sin^2 x = (1 - \cos 2x)$, we thus have:

$$Z = \frac{2h^2}{1 - \cos(2\theta - 2\alpha)} e^{i2\theta}$$

If now we put ψ for 2θ we have

$$Z = \frac{2h^2}{1 - \cos(\psi - 2\alpha)} e^{i\psi}$$

This means that the distance $2h^2 \div [1 - \cos(\psi - 2\alpha)]$ is to be laid off at the angle ψ. The polar equation will then be simply:

$$r = \frac{2h^2}{1 - \cos(\psi - 2\alpha)} \tag{6.6}$$

where r denotes the radius vector in the transformed field.

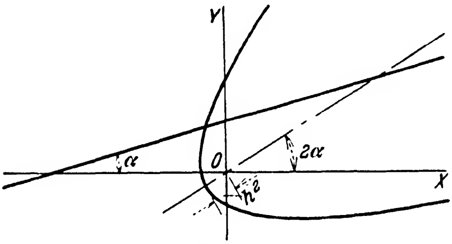

Fig. 64.

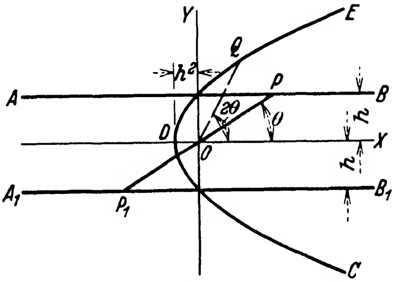

Fig. 65.

This is the equation to a parabola, the axis of which is inclined to X at an angle 2α, as indicated in Fig. 64.

The proof of this, as an exercise in analytical geometry, may be left to the interested reader.

However, the special case when the line is parallel to X may be followed a little further.

Equation (6.6) then becomes:

$$r = \frac{2h^2}{1 - \cos\psi} \tag{6.7}$$

This is readily put into rectangular coordinates as follows:

$$r = \frac{2h^2}{1 - x/r}$$

or

$$r = x + 2h^2$$

$$x^2 + y^2 = (x + 2h^2)^2$$

This gives

$$y^2 = 4h^2(x + h^2) \tag{6.8}$$

This is readily seen to be a parabola as in Fig. 65.

7. Special Conditions. In general, the relation between the two fields in conformal transformation is that of a one to one correspondence.

That is, one point of field 1 corresponds to one point of field 2 and *vice versa*. In special cases, however, two or more points of one field may correspond to one point in the other and *vice versa*. Thus with the transformation $Z = z^2$ as in Fig. 65, the line AB transforms into the parabola CDE. But the line $A_1 B_1$ will transform into the same parabola and thus two points in field 1 (as, for example P and P_1) will transform into the same point Q on the parabola. Thus let $|OP| = |OP_1| = r$ and let $POX = \theta$. Then OP will transform into a vector of length r^2 and angle 2θ, while OP_1 will transform into a vector of length r^2 and angle $2(\pi + \theta) = 2\pi + 2\theta$. But these two vectors are obviously the same.

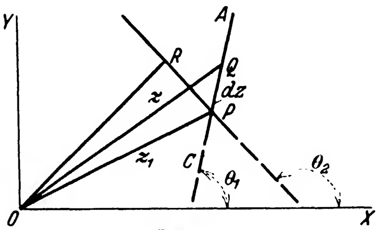

Fig. 66.

Again since $OP_1 = -OP$, this is simply an illustration vectorwise of the relation $(-z)^2 = (+z)^2$.

It is also evident that either half of the z plane (above or below the horizontal axis) will give the whole of the Z plane or *vice versa*, that the whole of the Z plane may be represented on either half of the z plane.

8. Singular Points. The transforming equation may take such a form that at a certain point or points the relation elsewhere between the two planes fails and the transformation is no longer conformal. Thus with the equation

$$Z = (z - z_1)^n f(z) \tag{8.1}$$

we have

$$\frac{dZ}{dz} = n(z - z_1)^{n-1} f(z) + (z - z_1)^n f'(z)$$

or

$$dZ = [n f(z) + (z - z_1) f'(z)] (z - z_1)^{n-1} dz \tag{8.2}$$

Now suppose that a line AC in the z plane is subjected to this transformation; (see Fig. 66). Let P be the point where $z = z_1$ and Q a nearby point. Then as a vector, $(z - z_1) = PQ$ and as Q approaches very near to P, $(z - z_1)$ and dz become the same. Likewise the entire expression within the bracket is a vector which, at the limit, as Q approaches very near to P, becomes simply $n f(z_1)$ [assuming $f'(z)$ finite]. This again is simply a vector with a scalar length which we may denote by r and an angle which we denote by α.

At a point very near P, remembering that here $(z - z_1) = dz$, we may, therefore, write (8.2) in the form

$$dZ = n f(z_1) (z - z_1)^n \tag{8.3}$$

But $(z - z_1)$ is the vector PQ with the angle θ_1 and $(z - z_1)^n$ is a vector with angle $n\theta_1$. Equation (8.3) thus shows dZ as the product of two vectors, one with the angle α and one with the angle $n\theta_1$. As Q approaches P in the z plane, therefore, P will approach the origin in the Z plane [see (8.1)] and Q will approach the same point at the angle $(\alpha + n\theta_1)$. Thus a line passing through P at an angle θ_1 in the z plane

will, in the Z plane, pass through the origin at an angle $(\alpha + n\theta_1)$. If then we should have another line RP passing through P at an angle θ_2, such line will, in the Z plane, pass through the origin at an angle $(\alpha + n\theta_2)$. The angle between these two lines in the z plane is $(\theta_1 - \theta_2)$ while in the Z plane it will be $n(\theta_1 - \theta_2)$. The angle has, therefore, been multiplied n fold by the transformation and the two fields are not conformal for the point P.

Points such as P, where the transformation is no longer conformal, are known as *singular* points.

It is, however, evident that if $n = 1$, the angle between the vectors on the Z plane will be the same as that on the z plane, and in such case the transformation will remain conformal.

In Fig. 67 let P denote again such a point. Let a small circle be described about P with radius a. Then the vector equation to such a circle is

$$z = z_1 + ae^{i\theta} \text{ [see V 11 } (f)].$$

or
$$z - z_1 = ae^{i\theta}$$

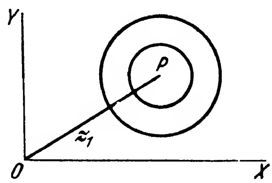

Fig. 67.

Then with the transformation of (8.1) if a is assumed very small $f(z)$ becomes $f(z_1)$ and we have

$$Z = f(z_1)\, a^n e^{in\theta}$$

This is a small circle about the origin on the Z plane with radius $a^n f(z_1)$ and traversed n times for once on the z plane. That is, if a tracing point be supposed to trace around the circle once on the z plane, the corresponding point on the Z plane will traverse its circle n times.

This is the same result as reached above when it was shown that the angle between any two radii to such a circle on the z plane would become n fold greater on the Z plane.

It should be noted, however, that the transformation for the circle itself is conformal—that is, a circle transforms into a circle. In fact, as may be seen, the conditions for conformal transformations are fulfilled in all respects *on* and *outside* of the circle about P. We may therefore define the area within which the conditions for conformal transformation are *not* fulfilled as the area within a small circle described about P as a center and since this circle may be made as small as we please, we may reduce to vanishing limits the parts of the area over which the transformation is not conformal.

Next suppose in (8.1) n negative so that the equation takes the form

$$Z = \frac{f(z)}{(z - z_1)^n} \tag{8.4}$$

Then similarly to (8.3) we shall have

$$dZ = \frac{f'(z_1)}{(z - z_1)^{(n-1)}} - \frac{nf(z_1)}{(z - z_1)^n} \tag{8.5}$$

In this case, as the point Q approaches P, $f'(z_1)/(z - z_1)^{(n-1)}$ will become an infinite vector with some angle β; $n f(z_1)/(z - z_1)^n$ will become an infinite vector with an angle $(\alpha - n\theta_1)$; the difference will, in general, be an infinite vector with angle depending on α, β, and the relative magnitudes of the two vectors. It cannot therefore be expressed as any simple multiple or function of α and β. The difference forming $d\,\boldsymbol{Z}$ will therefore be an infinite vector with angle having no direct relation to α, β, and θ_1. We cannot, therefore, express in terms of these angles, the angle of the infinite vector $d\,\boldsymbol{Z}$ at infinity. The same would hold for any other line RP and hence we can say nothing definite regarding the angle at ∞ in the Z plane between two lines such as QP and RP in the z plane. It is of interest to note that this equation applied to an indefinitely small region at P will transform it into an indefinitely large region at a great distance from P and such that angular relations at the point P in the z plane lose entirely their significance in the Z plane. The transformation is therefore non-conformal for the point P. A small area surrounding P may, however, be excluded from the transformation by the same device as in the preceding case. Outside of such area the transformation will remain conformal and thus all requirements of a practical case may be realized.

Next suppose the equation of the form

$$\boldsymbol{Z} = (z - z_1)(z - z_2)(z - z_3) \tag{8.6}$$

Then at any point on the z plane as located by z_1, z_2, z_3, etc., $\boldsymbol{Z}$ becomes zero and the point on the Z plane will come to the origin. It is then of interest to see, whether, in such case, the transformation remains conformal.

Suppose again, as in Fig. 66 that we have a point Q very near P, the point determined by z_1. Then (8.6) may be written

$$\boldsymbol{Z} = (z - z_1)\,\boldsymbol{Z}_1 \tag{8.7}$$

where $\boldsymbol{Z}_1$ is a vector representing the product of all the vector factors in (8.6) except $(z - z_1)$. Then comparing this with (8.1) it is seen that (8.7) is a special case of (8.1) where $n = 1$ and, as there noted, in such case the transformation is conformal for the point P.

If, however, the index of the factor $(z - z_1)$ should be n instead of 1, $(n > 1)$ the case becomes that of (8.1) and the transformation will be non-conformal at the point P.

Suppose again the transforming equation to take the form

$$\boldsymbol{Z} = \left(1 - \frac{z_1}{z}\right)\left(1 - \frac{z_2}{z}\right)\left(1 - \frac{z_3}{z}\right)\dots \tag{8.8}$$

Assuming n such factors, this is equivalent to

$$\boldsymbol{Z} = \frac{1}{z^n}(z - z_1)(z - z_2)(z - z_3)\dots \tag{8.9}$$

Then following the same line of reasoning as for (8.7) it follows that with any equation of the form of (8.8) or (8.9), the point P on the z plane for $z = z_1$, z_2, etc., will, on the Z plane, come to the origin, but the transformation for and at all such points will remain conformal.

If, however, (8.8) contains a factor with index n $(n > 1)$, then for the same reasons as with (8.6) the transformation for such point becomes non-conformal.

Again problems of transformation may arise involving equations for which the derivative $d\,Z/d\,z$ has the form

$$\frac{d\,Z}{d\,z} = (z - z_1)\,(z - z_2)\,(z - z_3) \tag{8.10}$$

Then referring again to Fig. 66, with reference to the point P we may write (8.10) in the form

$$d\,Z = Z_1\,(z - z_1)\,d\,z$$

where Z_1 denotes the product of all the vectors other than $(z - z_1)$. But near the limit $(z - z_1)$ and $d\,z$ are the same so that we have finally, for a point very near P, $\quad d\,Z = Z_1\,(z - z_1)^2 \quad$ (8.11)

Referring to (8.3) it is seen that (8.11) is a special case of (8.3) $(n = 2)$ and hence, as with (8.3) $(n > 1)$ the transformation will here be non-conformal.

Suppose again, parallel to (8.8) that we have

$$\frac{d\,Z}{d\,z} = \left(1 - \frac{z_1}{z}\right)\left(1 - \frac{z_2}{z}\right)\left(1 - \frac{z_3}{z}\right)\ldots. \tag{8.12}$$

Then as with (8.9)

$$\frac{d\,Z}{d\,z} = \frac{1}{z^n}\,(z - z_1)\,(z - z_2)\,(z - z_3)\ldots. \tag{8.13}$$

Following the same strategy as with (8.10), this may be written,

$$d\,Z = \frac{Z_1}{z^n}\,(z - z_1)\,d\,z$$

and for a point Q very near P, this becomes

$$d\,Z = \frac{Z_1}{z_1{}^n}\,(z - z_1)^2 \tag{8.14}$$

This again is a special case of (8.3) $(n = 2)$ and hence for such point the transformation will be non-conformal.

With these cases, as with the preceding, wherever the transformation is non-conformal at a given point such as P, the small area so affected may always be excluded from the region as a whole, by the device of the small circle as in the cases of (8.1), (8.4).

From the preceding discussion it will be evident that for any form of transforming equation such as $Z = z^n$ or $z^n f(z)$, the origin will be a singular point.

Thus for the transformation $\boldsymbol{Z} = z^2$, the origin is a singular point as may be noted in Fig. 65 where, on the z plane the angle $XOY = 90^0$ becomes on the Z plane the angle $XOX = 180^0$.

If in an equation of the form (8.1) the index be taken as $1/n$ instead of n, the same general formulae and results will apply. In this case, however, two lines on the z plane meeting at P at an angle θ, will, on the Z plane pass through the origin at an angle θ/n. This fact becomes of importance in connection with the application of the method of conformal transformation to certain problems in aerodynamics.

As noted, all such points where the conditions of conformal transformation no longer hold, are known as *singular* points and the conditions for the existence of such a point may be seen to be the presence of a zero factor in the expression for $\boldsymbol{Z}$, such factor having an index n greater than 1, or otherwise the existence of a zero factor in the expression for $d\,\boldsymbol{Z}/d\,z$, such factor having an index greater than zero.

DIVISION B

FLUID MECHANICS, PART I

By

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PREFACE

The treatment of the subject of Fluid Mechanics in the present Division is intended to furnish a relatively simple and "easy reading" introduction to those parts of the subject of special interest to the student of aerodynamics. For the proper understanding of the various monographs which make up this presentation of the general subject of Aerodynamic Theory, there will be needful some familiarity with the subject of Fluid Mechanics beyond, or at least somewhat different from that gained by the student in the courses found in most of our technical colleges and schools. To many others, to whom these matters have been familiar at one time, facility, through disuse, may have been lost.

For both classes of readers, this treatment of the subject is presented in the hope that it will furnish close at hand such help in the elements of fluid mechanics as will be most often needed in the reading of the later Divisions of the work.

Familiarity is assumed with ordinary college mathematics together with certain of the subjects discussed in Division A.

For more advanced readers and for certain phases of the subject, the treatment in Division C will be found of special interest and value. There will be noted a slight overlapping as between the treatments in Divisions B and C in the subject of three-dimensional fields of flow. In Division C the treatment is, in general, three-dimensional; in Division B, it is, for the most part, two-dimensional. However, a sufficient discussion of three-dimensional fields has been included to give to the reader who might find somewhat difficult the treatment of Division C, at least an introduction to the character and fundamental equations of three-dimensional fields of fluid motion.

Standard text books on the subject and including Lamb's classical treatise, have been freely drawn upon for suggestions and for special points in the treatment.

CHAPTER I
FUNDAMENTAL EQUATIONS

1. Introductory, Characteristics of a Fluid. In order to develop any approach to a mathematical treatment of a fluid in motion or of the force reaction between a fluid and solid bodies with which it may be in contact or dynamic relation, there must first be formed some picture of the physical constitution of the fluid, or at least of its physical characteristics. This picture may again relate to an actual fluid as occurring in nature or to an ideal medium with specifications permitting a more extended mathematical treatment than is possible with fluids as they actually are.

With actual fluids, the following five physical characteristics will, in general, be involved in the problems of fluid mechanics.

Pressure	*Temperature*
Volume	*Density*
	Viscosity

Of these terms, only the last will perhaps need some explanation. The term *viscosity* is given to that property of fluids as a result of which they present some resistance to shearing, that is, to the sliding movement of one particle past or near another. Thus in liquids like tar or heavy oil, the viscosity is large; in water it is relatively small; in gases like air it is also small but not negligible, or at least only so for low relative velocities of adjacent particles.

Further reference to viscosity as a factor in fluid motion will be made at a later point and it will be sufficient here to note simply that since viscous forces vary directly with the relative velocity of sliding between adjacent particles or parts of the fluid, such forces, when the relative velocities are low, may be so small as to be negligible. When the relative velocities are large, however, the viscous forces may become so large as to dominate the situation and to bring about an entirely different set of conditions from those pertaining to small or vanishing viscosity.

The attempt to introduce the influence of viscosity into the equations of fluid mechanics, however, results in such complication that up to the present time, no general method for the treatment of such equations has been found. In order, therefore, to make possible some measurably extended treatment of the subject, an ideal medium is assumed in which the viscosity is zero, or in any event is negligible for the relative velocities which are involved.

Of the remaining characteristics, density must always be retained and included, since in fact all dynamic and static effects with fluids vary directly with the density. Furthermore, for any given mass of fluid, volume and density, of course, vary inversely and for all actual

fluids, volume, pressure and temperature are interrelated in such manner that any two being given, the third is fixed. With any fluid it is possible to form an algebraic equation connecting together these three characteristics of pressure, volume and temperature, and to a degree of accuracy closely agreeing with the facts of experience, at least over limited ranges of change for these quantities. However, any attempt to introduce such a relation into the equations of fluid motion leads again to complications which limit or greatly hamper a broad introductory treatment of the subject. It results that for the purposes of such a treatment it is convenient to assume the density as independent of the consequences of a change of pressure and temperature.

The term *fluid* implies, moreover, two classes of substances—liquids and gases. For the former, density (or volume) is affected only in small degree by changes in pressure and temperature and for such changes as are liable to arise in connection with the usual problems of fluid mechanics as applied to liquids, we may, without sensible error, assume the density (or volume) as independent of the results due to changes in pressure and temperature, and therefore, as constant throughout the course of the problem.

With gases, or gases and vapors, however, the case is quite different. The variation of volume (or density) with changes in pressure and temperature is much more notable.

For the so-called permanent gases, such as oxygen, nitrogen, the air as a mixture of the two, hydrogen, helium, etc., it is well known that an equation of the form

$$\frac{pV}{T} = \text{constant} \tag{1.1}$$

(where p = pressure, V = volume, and T = absolute temperature) very closely represents the facts of experience. That is, for such fluids, the volume varies inversely with the pressure or the density directly with the pressure, the temperature remaining constant, or the volume varies directly and the density inversely with the absolute temperature, the pressure remaining the same. However, in actual problems no one of these characteristics is likely to remain constant, but rather all are liable to change according to some law which may prevail for the problem in question. Thus in many cases the adiabatic law of pressure volume change might hold, expressed by the equation

$$pV^{\gamma} = \text{constant}$$

where γ as an exponent is very near to 1.4. Such an equation for p and V combined with (1.1) will give, as may be desired, relations between p and T or between V and p for such a path or history of change.

As before noted, however, it is found that the attempt to introduce even the simplest of these relations into the equations for fluid motion

will seriously hamper the desired development of the subject and for present purposes, therefore, we assume the volume and hence the density to be constant; or otherwise, that such changes in pressure and temperature as may actually occur are too small to sensibly affect the density of the medium, at least in comparison with the other factors in the problem.

Thus finally we arrive at the *ideal fluid* which is used as a hypothetical medium for the development of the equations of flow and for the investigation of various problems arising out of the fields of flow over which these equations give us control. This fluid is then characterized by what is left of the above five characteristics when we assume:

(1) Viscosity = zero.

(2) Density (or volume) = constant.

The temperature disappears entirely, since in itself, it is not a characteristic of the mechanics of the medium, and since in any event we assume its influence on the other characteristics to be constant. The pressure remains, since it is a characteristic of the mechanics of the fluid. We have simply assumed that such *changes* in pressure as may occur will not sensibly affect the density.

With regard to the justification for such departures from actual conditions, it will be sufficient to note here that the assumption of zero viscosity is justified by the results which it gives, representing, as it does, a condition which is the limit toward which actual fluids may approach and to which, under conditions of small relative velocity between adjacent particles, they may sensibly attain. On the other hand, due caution must be exercised in the use of the results based on the assumption of a zero or vanishing viscosity, especially where the relative velocity of adjacent particles is high. In such case, as noted above, the results based on the assumption of zero viscosity will give no proper account of the actual phenomena. In Division G of this work will be found a special treatment of the subject of viscous fluids with special reference to the influence of viscosity on the problems of chief importance in aeronautics.

With regard to the assumption of constant density, note has been already made of the fact that for liquids such assumption is generally quite justified by the almost inconsiderable change in density for very considerable changes in pressure. Thus for example, with a given volume of water, an increase in the pressure, from 15 pounds to 30 pounds per square inch will result in a decrease in volume of only about 1 part in 20,000.

With air, on the other hand, such an increase in the unit pressure, the final temperature being the same as the initial, will cause a reduction of 1 part, in 2. However, experience with the problems of airflow in

the midst of a mass of air of indefinite extent, such as we are concerned with in the domain of aeronautics, shows that only under extreme conditions are the variations in pressure sufficient to produce any sensible change in the density. In cases where the velocity of flow or the velocity of a body passing through the air approaches the velocity of sound in air (1100 ft./sec.) the changes in pressure will be sufficient to sensibly modify the density and any attempt at a full treatment of such problems must include some recognition of this fact. However, most of the problems of aerodynamics involve velocities well within this limit and in any event it is advantageous to establish the general equations and principles of fluid mechanics on the assumption of density constant, or otherwise flow as of a medium incompressible under the changes of pressure incident to the motion. In Division H of this work will be found a treatment of the subject of compressible fluids with special reference to the influence of this feature on the problems of chief importance in aeronautics.

With this general discussion regarding the characteristics of the medium with which we are to deal, we may now proceed with the development of a foundation upon which to erect the mathematical expression for the conditions of the problems with which we are concerned.

The distinguishing characteristic of this ideal medium, as already noted, is perfect mobility or an absence of any resistance to a shearing movement. From this it results that the pressure at any point in such a fluid will be exerted equally in all directions. It also follows that there can be no tangential force between a fluid and the surface of a solid with which it is in contact and hence that the total force reaction between a fluid and any small element of a surface with which it is in contact, must be normal to the surface at that point.

2. Physical Conditions, Notation. Having thus defined the character of the fluid medium with which we are to deal, we must next picture a general set of physical conditions—a physical aggregate as it were—within which will be found the phenomena forming the subject of our study. This aggregate we picture in the general case of an indefinite fluid medium within which are solid bodies or rigid guiding surfaces, and with relative motion between the solid and fluid members of such a system. We have next to picture a set of rectangular axes, X, Y, and Z to which we may refer the various physical quantities with which we shall be concerned. These may be taken arbitrarily in any direction whatever. Where suitable, however, it is customary to take the axis of X parallel to the direction of relative motion as it exists far distant from the solid bodies placed within the medium, and the axes of Y and Z conveniently at right angles to X and to each other. As a physical picture it is often possible to assume the relative motion horizontal in which case X will be taken as above, Y horizontal, $\perp$ to X and Z vertical.

Again, motion and velocity are always relative and so here some datum must be chosen relative to which these quantities may be measured. Thus we may identify ourselves with a frame of reference attached to a solid body placed in the field and consider the motions and velocities within the fluid medium as it flows past our reference frame. Or again, we may identify ourselves with a frame of reference considered as attached to the great body of fluid far distant from any solid body, and consider the latter as it moves through such a medium, together with the local movements and velocities within the fluid set up by the moving body. For most purposes with which we shall be concerned in aerodynamic problems, the former of these will be the more convenient.

We may therefore set up a notation as follows:

X, Y, Z are the axes defined above.

x, y, z are the coordinates of any point in space.

u, v, w are the component velocities in the directions respectively of X, Y, Z.

V is resultant or total velocity.

t is time reckoned from any convenient datum.

ϱ is density of medium.

p is pressure at any point x, y, z.

Then by definition

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} \\ v &= \frac{\partial y}{\partial t} \\ w &= \frac{\partial z}{\partial t} \end{aligned}\right\} \tag{2.1}$$

Now in the general case of fluid movement we must consider the fluid as changing in velocity from point to point in space at any one instant of time, and from moment to moment of time at any one point in space. That is, the component velocities u, v, w, together with the resultant velocity V must be considered as functions of both space and time. The complete derivative of any component velocity such as u must therefore be taken as the sum of the partial derivatives relative to the four variables x, y, z, t. Thus

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

Substituting from (2.1) this gives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{2.2}$$

with similar expressions for $\dfrac{dv}{dt}$ and $\dfrac{dw}{dt}$.

However the equations resulting from this general assumption can only be treated in special cases and for the simpler and introductory phases of the subject the further assumption is made of a state of *steady motion.*

That is, we assume that at any one point in space, the velocity conditions within the fluid remain constant in time, and vary only as we pass from one point in space to another. We thus reach the final simplified case where we are concerned only with the distribution of velocity (in direction and magnitude) throughout the field of space involved in the problem.

Equations of the form of (2.2) thus become reduced to

$$\left.\begin{aligned} \frac{du}{dt} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \frac{dv}{dt} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \frac{dw}{dt} &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \qquad (2.3)$$

The first of these equations is thus to be interpreted as giving the time rate of change in the component velocity u found in passing from an initial point P to a near by point P_1, the coordinates of P_1 relative to P being dx, dy, dz, and with a similar interpretation for the other two equations.

3. A Field of Fluid Flow as a Vector Field. It is clear that a field of steady fluid flow in every way meets the conditions for representation by a vector. That is, the essential characteristic at any point in the field is motion in a certain direction and with a certain velocity. It follows that the entire content of Division A VI on vector fields is directly applicable to such fields of fluid flow. These various results will therefore be assumed as applying to such fields of flow and need not be here repeated. It may be well to note, however, that the type of fluid motion which is here contemplated, and to which, in particular, the results of this chapter apply, is one meeting the conditions:

(1) A perfect fluid (viscosity zero).
(2) Steady motion.
(3) Constant density or volume (incompressible fluid).
(4) Irrotational motion.

The treatment of vector fields, however, is wholly geometrical in character. Dynamic elements are lacking. No such limited character of treatment can therefore serve fully for the study of a field of fluid motion. Factors involving mass, force and energy must be introduced, and to this end we must, in some way, involve certain essential properties or characteristics of the fluid in relation to the space in which it moves. There are two such properties or characteristics available, one leading to the so-called equation of continuity, while the other has for its expression the basic relation in mechanics between force, mass, and acceleration.

4. The Equation of Continuity. This equation is based on the fact that if we consider any given volume in space bounded by a closed

geometrical surface, then at any instant of time, the flow of fluid across this boundary surface from without inward must equal that from within outward. Or algebraically, if inflow be taken as positive and outflow negative, the total flow of fluid across such a closed boundary must be zero. It is clear that if this condition were not fulfilled, there would be continuous increase or decrease in the amount of fluid within such volume and this would violate the conditions of steady flow, and likewise those of constant density.

The expression of this condition in mathematical form [Division A VI (6.2)] is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.1}$$

This is known as the equation of *continuity* and simply expresses in mathematical form for a small element of volume (and hence for any closed space) the physical fact as stated above.

5. The Equation of Force and Acceleration. The second equation to be developed is a statement in mathematical form that the force required to produce an acceleration in any small element of the fluid is measured by the product of the mass and the acceleration produced.

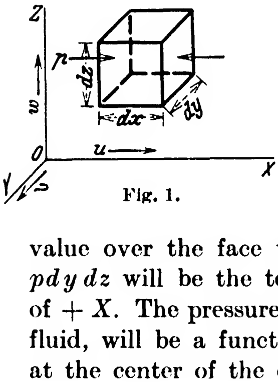

Fig. 1.

Taking again a small element of volume as in Fig. 1 let p be the pressure at the center of the face $dy\,dz$ nearer the origin. Then while the pressure must be assumed as variable over this face, the error involved in taking p as the mean value over the face will be of the second order and negligible. Hence $p\,dy\,dz$ will be the total pressure on this face acting in the direction of $+X$. The pressure p, variable throughout the space occupied by the fluid, will be a function of the coordinates x, y, z; and for the pressure at the center of the outer face $dy\,dz$ we shall have

$$\left(p + \frac{\partial p}{\partial x}\,dx\right)$$

Then the total pressure on this outer face and acting in the direction of $-X$ will be

$$\left(p + \frac{\partial p}{\partial x}\,dx\right) dy\,dz$$

The net total pressure acting on the element in the direction of $+X$ is therefore:

$$-\frac{\partial p}{\partial x}\,dx\,dy\,dz$$

Now suppose that there is some force other than pressure acting on the fluid and let P, Q, and R be the components of such force per unit mass taken parallel respectively to X, Y, and Z. Then

$$\left(P\varrho - \frac{\partial p}{\partial x}\right) dx\,dy\,dz$$

will be the total force acting on the element of the fluid along the direction of $+X$. The acceleration in this direction is du/dt and the mass is $\varrho\,(dx\;dy\;dz)$. Hence we shall have the equation:

$$\left(P\varrho - \frac{\partial p}{\partial x}\right)(dx\,dy\,dz) = \varrho\,(dx\,dy\,dz)\,\frac{du}{dt}$$

or

$$P - \frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{du}{dt} \tag{5.1}$$

The general value of du/dt from (2.2) then gives,

$$P - \frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{5.2}$$

with similar equations for the total accelerations along Y and Z.

For the applications of fluid mechanics to the problems of aeronautics (there are many other applications as well) we may, in many cases, assume the absence of forces such as those represented by the components P, Q, and R. In such case we shall have the set of equations:

$$\left.\begin{aligned}
-\frac{1}{\varrho}\frac{\partial p}{\partial x} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\\
-\frac{1}{\varrho}\frac{\partial p}{\partial y} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\\
-\frac{1}{\varrho}\frac{\partial p}{\partial z} &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\end{aligned}\right\} \tag{5.3}$$

Eqs. (4.1) and (5.3) may be considered as the basic equations of fluid mechanics under the restricted conditions assumed, *viz.*

(1) Fluid without viscosity.
(2) Density constant.
(3) Steady motion.
(4) No external accelerating forces.

These equations, however, in this form, are not readily applicable to specific cases and in order to render them more directly useful, some working over is required. To this end we proceed as follows:

6. Bernoulli's Equation. We have already assumed the pressure p to vary, at any one instant, throughout the space occupied by the fluid. This means that along with the velocities, pressure must also be considered as a function of the coordinates x, y, and z. The complete differential of p will therefore be of the form:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \tag{6.1}$$

But comparing this with Eqs. (5.3) it is clear that if we multiply the latter respectively by dx, dy, dz we shall have, on the left, multiplied by (—) $1/\varrho$, the respective partial differentials of p with reference to x, y, and z. Thus,

$$\left.\begin{aligned}
-\frac{1}{\varrho}\frac{\partial p}{\partial x}\,dx &= u\frac{\partial u}{\partial x}\,dx + v\frac{\partial u}{\partial y}\,dx + w\frac{\partial u}{\partial z}\,dx \\
-\frac{1}{\varrho}\frac{\partial p}{\partial y}\,dy &= u\frac{\partial v}{\partial x}\,dy + v\frac{\partial v}{\partial y}\,dy + w\frac{\partial v}{\partial z}\,dy \\
-\frac{1}{\varrho}\frac{\partial p}{\partial z}\,dz &= u\frac{\partial w}{\partial x}\,dz + v\frac{\partial w}{\partial y}\,dz + w\frac{\partial w}{\partial z}\,dz
\end{aligned}\right\} \tag{6.2}$$

Comparing again with (6.1), it is clear that adding these equations we shall have, on the left, a complete differential which will immediately integrate to $-\frac{1}{\varrho}p$. On the right hand side, however, are several terms which do not admit of direct integration in the form in which they are. To transform these we go back to (2.1), where, dividing by pairs we find:

$$\frac{\partial x}{\partial y} = \frac{u}{v}, \qquad \frac{\partial x}{\partial z} = \frac{u}{w}, \qquad \frac{\partial y}{\partial z} = \frac{v}{w},$$

whence

$$\begin{aligned}
v\,dx &= u\,dy \\
w\,dx &= u\,dz \\
w\,dy &= v\,dz
\end{aligned}$$

Substitution of these relations in (6.2) then gives,

$$\left.\begin{aligned}
-\frac{1}{\varrho}\frac{\partial p}{\partial x}\,dx &= u\frac{\partial u}{\partial x}\,dx + u\frac{\partial u}{\partial y}\,dy + u\frac{\partial u}{\partial z}\,dz \\
-\frac{1}{\varrho}\frac{\partial p}{\partial y}\,dy &= v\frac{\partial v}{\partial x}\,dx + v\frac{\partial v}{\partial y}\,dy + v\frac{\partial v}{\partial z}\,dz \\
-\frac{1}{\varrho}\frac{\partial p}{\partial z}\,dz &= w\frac{\partial w}{\partial x}\,dx + w\frac{\partial w}{\partial y}\,dy + w\frac{\partial w}{\partial z}\,dz
\end{aligned}\right\} \tag{6.3}$$

It is clear that, thus written, (6.3) on the right form respectively the complete differentials of $u^2/2$, $v^2/2$, $w^2/2$, considered as functions of x, y, and z.

Hence we may write the sum in the form

$$\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)\,dx + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2)\,dy + \frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dz$$

or

$$\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx + \frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy + \frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz$$

Hence, for the sum of (6.3), integrating on both sides, we shall have, for the change between two sets of conditions denoted by 0 and 1

$$-\frac{p}{\varrho}\Big]_0^1 = \frac{V^2}{2}\Big]_0^1 \tag{6.4}$$

In words, this equation states that between any two points in the space occupied by the fluid, the decrease in the value of p/ϱ is equal to the increase in the half square of the velocity, or *vice versa*. Again putting in the limits we have an equation of the form:

$$\frac{p_0}{\varrho} + \frac{V_0^2}{2} = \frac{p_1}{\varrho} + \frac{V_1^2}{2} \tag{6.5}$$

or

$$p_0 + \frac{\varrho V_0^2}{2} = p_1 + \frac{\varrho V_1^2}{2} \tag{6.6}$$

The general expression $p + \frac{\varrho V^2}{2}$ is called the total pressure, made up of p the *static* or *actual pressure*, and $\varrho V^2/2$, the *dynamic pressure*. In this equation if ϱ is measured in standard units of mass (pound or kilogram), then p must be measured in absolute units of pressure. If the ordinary engineering or gravity units of force are used—that is, if p is measured in terms of pounds or kilograms of force per unit area—then ϱ must be considered as measured in terms of a unit of mass g times larger than the standard unit. This will reduce the measure of ϱ in the ratio g and will thus measure the dynamic pressure also in gravity units. Or otherwise if the measure of ϱ is kept in standard units, we may reach the same result by writing the equation

$$p_0 + \frac{\varrho V_0^2}{2g} = p_1 + \frac{\varrho V_1^2}{2g} \tag{6.7}$$

With this definition of ϱ, we may also revert to the form

$$\frac{p_0}{\varrho} + \frac{V_0^2}{2g} = \frac{p_1}{\varrho} + \frac{V_1^2}{2g} \tag{6.8}$$

In this form p/ϱ (with ϱ considered as weight of unit volume) together with $V^2/2g$ will both have the dimensions of a length. The first will represent the height of a column of fluid sufficient to give a pressure p at the base while the second is the gravity head corresponding to the velocity V. The sum of these will thus represent a height, often represented by H and called the total head. Thus

$$H = \frac{p}{\varrho} + \frac{V^2}{2g} \tag{6.9}$$

Equations (6.5) . . . . . (6.9), represent in terms of one unit or the other, *Bernoulli's equation*, a relation of great importance in the study of the problems of fluid mechanics.

The Bernoulli Constant in a Field with Rotation. From the preceding discussion it is clear that the Bernoulli equation is primarily the equation to a single line of flow. It asserts the conservation of energy along such a line or small tube of flow. It is of interest, however, to consider the consequence of a change in the total head or pressure (representing energy) in passing from one line of flow to another.

Thus (6.6) may be written in the form

$$p + \frac{\varrho V^2}{2} = P$$

where P represents the total pressure as there defined. Denoting the direction normal to the line of flow by N and taking derivatives along N we have

$$\frac{\partial p}{\partial N} + \varrho V \frac{\partial V}{\partial N} = \frac{\partial P}{\partial N} \tag{6.10}$$

Assuming that P is not a constant from one line to another, this gives its rate of change along the normal, made up as indicated by the two terms of the equation.

In general the line of flow will be curved. Let R denote the radius of curvature at any given point. Then an element bounded by two near by lines of flow, two near by radii and of unit thickness, will have the volume $R d\theta dR$ or $R d\theta dN$. The centrifugal force on this element will be $\varrho R d\theta dR \cdot V^2/R = \varrho dR d\theta \cdot V^2$. The pressure p acting on the two sides dR of the element will have a radial resultant $p dR d\theta$. These two forces will then be responsible for the increment of pressure load between the inner surface $R d\theta$ and the outer surface $(R + dR)d\theta$. The unit pressure on the former is p, on the latter $(p + \frac{\partial p}{\partial R} dR)$. Taking in this way the increment of pressure load on the outer face of the element, equating the result to the sum of the expressions above for centrifugal force and radial resultant, we find, after reduction,

$$\frac{\partial p}{\partial R} = \frac{\partial p}{\partial N} = \frac{\varrho V^2}{R} \tag{6.11}$$

combining (6.10) and (6.11) we have

$$\frac{\partial P}{\partial N} = \varrho V \left(\frac{V}{R} + \frac{\partial V}{\partial N}\right) \tag{6.12}$$

Reference to Division A VI (8.3) then gives

$$\frac{\partial P}{\partial N} = 2\,\omega\,\varrho\,V \tag{6.13}$$

This result indicates that in a field with rotation the value of P will change from point to point in the field across the line of flow; and *vice versa* in a field without rotation, the value of P will be the same throughout the field.

7. A Field of Flow; A Tube of Flow; Conditions of Equilibrium of a Field Within a Portion of a Tube of Flow; Momentum Theorem. The terms Field and Field of Flow have already been employed. They are plainly intended to include the entire ensemble of phenomena presented by a mass of fluid between which and certain constraining boundaries or solid bodies, there is relative motion. To fix the ideas, we now assume axes of reference fixed relative to the solid bodies or boundaries, with the fluid streaming by. If we fix our attention on what we may consider as the ultimate particles of the fluid, we shall find these particles all moving in various directions in the field. At any one point, however, the motion is definite and, in general, single valued. Now assume a line in the field such that its tangent at every point is in the direction of the fluid movement at that point. Such a line is called a stream-line.

The distinction between stream-lines and particle paths and their dependence on the system of axes of reference will be discussed in later sections. For the present the above definition, with the axes as assumed, serves to define a stream-line which is at the same time a particle path or line of flow.

Suppose now a closed curve lying in the field of flow—ABC Fig. 2. There must be a stream-line touching every point on this curve, and the ensemble of these lines will form a closed tube along and through which flows the fluid which passes within the curve. This defines a tube of flow. See also Division A VI 5. It is clear that the fluid which at one time or point is flowing within this tube will always and at all points flow within the tube. It thus marks out a definite part of the fluid which may thus be considered as an entity, separate from the remainder of the field. In fact we may imagine the stream-lines forming the surface of the tube to be replaced by a smooth rigid surface of the same geometrical form, thus giving a rigid tube within which the fluid may be supposed to move as a fluid system in itself. This general concept of the replacement of a line or tube of flow by a smooth rigid line or surface boundary is of frequent use in the study of the problems of fluid mechanics, as will appear in subsequent sections.

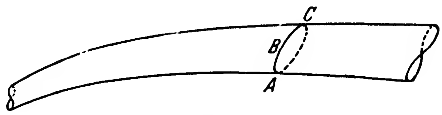

Fig. 2.

We have now to establish an important theorem with regard to the total force reaction between the fluid and the tube for any part of such a tube of flow. The same theorem will, of course, apply to the case of a rigid pipe or conduit, or any element or part thereof, within which a fluid is flowing.

Let $A\,B$ Fig. 3 be a portion of a tube of flow. The fluid in flowing from A to B may and in the general case, does undergo a change in momentum. This change must, of course, be considered in its vector sense, that is, both as to change in magnitude and in direction. But a change of momentum is the evidence of the action of a force and as we know, the time rate of such change is a measure of the force and the direction of the change is the direction in which the force is applied. In a vector sense, if PQ represents the momentum per unit time passing the entrance A and PR that which passes the exit B, then in a vector sense, QR represents the change in the momentum per unit time, and will therefore represent as a vector the direction and magnitude of the resultant force acting on the fluid within the conduit between A and B.

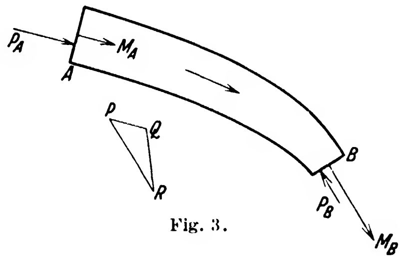

Fig. 3.

Now neglecting gravity, the influence of which is negligible in the cases with which we are usually concerned, such force can be applied to the fluid only through:

(1) Pressure over the surfaces A and B, both acting inward.

(2) Pressure acting from the inner surface of the tube *upon* the fluid.

It follows that the vector *sum* of these three forces must equal the vector *difference* in the two rates of momentum flow at A and at B.

Let P_A = pressure on face A directed in.
P_B = pressure on face B directed in.
P_T = resultant pressure between surface of tube and fluid directed from the former to the latter.
M_A = momentum entering at A per unit time.
M_B = momentum issuing from B per unit time.

Then considering these all in a vector sense, we shall have a vector equation as follows:

$$\boldsymbol{P}_A + \boldsymbol{P}_B + \boldsymbol{P}_T = \boldsymbol{M}_B - \boldsymbol{M}_A$$

Now when the conditions of flow are known, we shall know all terms in this equation with the exception of $\boldsymbol{P}_T$. Whence we may write

$$\boldsymbol{P}_T = \boldsymbol{M}_B - \boldsymbol{M}_A - \boldsymbol{P}_A - \boldsymbol{P}_B \tag{7.1}$$

This may be written

$$\boldsymbol{P}_T = \boldsymbol{M}_B + (-\boldsymbol{M}_A) + (-\boldsymbol{P}_A) + (-\boldsymbol{P}_B) \tag{7.2}$$

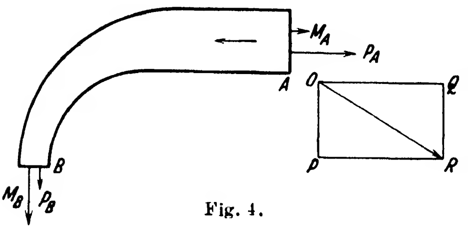

Fig. 4.

But a vector with the negative sign means simply the reversal of the vector in direction. This equation, therefore, directs the combination, as a sum, of the following vectors:

The issuing momentum directed *outward.*

The entering momentum directed *outward.*

The pressure over the faces A and B both directed *outward.*

The vector sum will then give the vector value of P_T, the resultant force acting from the walls of the tube upon the fluid. The reverse action—that of the fluid upon the tube, will be the reverse of this vector.

The relation shown by (7.1) or (7.2) is often known as the "momentum theorem".

Thus for illustration, suppose an elbow turn as in Fig. 4 with reduction in size from A to B. Then the various vectors will be laid off as indicated and making $OP = M_B + P_B$ and $OQ = M_A + P_A$, we shall have OR as the resultant in magnitude and direction for the force acting from the walls of the elbow *on* the fluid and RO as the resultant force acting from the fluid *on* the elbow.

8. Impulse and Impulsive Forces. Many problems in the domain of fluid mechanics can be advantageously treated on the assumption that a given or assumed state of motion or a change in such state has been brought into existence through the action of a suddenly applied or impulsive field of force. In mechanics a force is defined as the time rate of change of momentum; or in symbols:

$$F = \frac{dM}{dt}$$

whence,
$$dM = F\,dt$$

The product $F\,dt$, or its equal dM, is then defined as the measure of an *impulse*. If we denote an impulse by the symbol P_i, we shall then have
$$P_i = F\,dt \tag{8.1}$$

In this equation it is obvious that dt may be very small and F correspondingly large; and at the limit dt may denote a period of time indefinitely small and F a force indefinitely large. In considering the impulse which is assumed to have brought about some change in a state of fluid motion, we are not concerned with the individual magnitudes of F and dt, but only with their product represented by the change of momentum produced. As a mechanical magnitude, therefore, impulse is of the nature of momentum and specifically is measured by the change of momentum characterizing the two states of motion, before and after the application of the impulse. The chief uses of this concept are found in its application to an existing state of motion and in the assumption that such state has been created from a state of rest by the application of an impulse field.

Referring to Eq. (5.2) let us multiply through by dt and then integrate between limits 0 and dt. During this very short period of time the force P, here assumed very great may be assumed constant and the integral will result as the impulse $P\,dt = P_i$. Denoting similarly the impulse due to the pressure at the point x, y, z by ω (called the pressure impulse) we shall have, for the second term, $\frac{1}{\varrho}\frac{d\omega}{dx}$. The first term on the right gives du or here $\triangle u$ (since the change in velocity is assumed finite) and all the remaining terms become zero since the products of the finite velocities u, v, w by the time interval dt will vanish in comparison with the other terms.

This gives finally the equation
$$P_i - \frac{1}{\varrho}\frac{d\omega}{dx} = \triangle u \tag{8.2}$$
with similar equations for the components along Y and Z. These equations will then give the measure of the component changes in velocity due to the application of a system of impulsive forces as assumed.

Again assume only the existence of the impulse pressure ω and the production from a state of rest of a velocity u. Equation (8.2) then reduces to
$$\frac{d\omega}{dx} = \varrho u$$
and if we assume the motion when produced to have a potential φ, we shall have $u = d\varphi/dx$ and this will give
$$\frac{d\omega}{dx} = \varrho\frac{d\varphi}{dx}$$
or
$$\omega = \varrho\varphi \tag{8.3}$$

That is, at any point, the impulsive pressure, as defined above, is measured by the product of the density and the velocity potential at that point.

9. Energy of the Field in Terms of Velocity Potential. The concept of a field of fluid motion as generated by the action of an impulse furnishes a ready means for measuring the energy of a field of flow. Obviously if we are concerned with the flow of an indefinite field past a stationary obstacle, the energy will be infinite. If, however, we consider the inverse problem of an object moving in an indefinite field, the energy of the field associated with such movement becomes finite. Thus suppose, for example, an indefinite field of fluid in which we take a given direction as the axis of X and on which we place a body of any given form. Then suppose in time dt an impulsive movement to be impressed on the body, in a direction along the axis of X, and changing its velocity from zero to V. Then following, assume the motion to continue without further change. We shall then have a field of flow as produced by the steady motion of the body moving with the velocity V in an indefinite field of fluid, considered in itself as originally at rest throughout the extent of the field.

As a simple illustration, we may imagine a two-dimensional field with a straight line moving at right angles to the axis of X. Such a field will be discussed in further detail in a later section.

It is of interest to note, at this point, that the fluid medium assumed for mathematical purposes is incompressible. In such a medium the propagation of pressure changes is instantaneous and hence the field distribution of p_i, as in 8, will be established coincidently with the movement of the generating body. Hence the establishment of the impulse field may be conceived of as coincident with, and as the direct consequence of, the movement of the generating body.

Now it is evident that the energy in such a field of motion can have no source other than the work which is expended upon the field by the generating body.

Let p_i be the impulse pressure at a given point on the surface of the generating body. This, acting normal to the surface, and in time dt generates a velocity $\partial\varphi/\partial n$. The initial velocity is zero and the accelerating force is uniform. Hence the mean velocity is $(1/2)\,\partial\varphi/\partial n$ and in time dt the distance moved will be

$$\frac{1}{2}\,\frac{\partial\varphi}{\partial n}\,dt$$

This is evidently the distance through which the force p_i is effective and hence the work due to an element of the surface dS of the generating body will be the product of the two. But this will measure an element of the energy E of the field and hence we have

$$dE = \frac{1}{2}\,p_i\,\frac{\partial\varphi}{\partial n}\,dt\,dS$$

But $p_i\,dt = \omega = \varrho\varphi$ and hence finally (irrespective of algebraic sign)

$$dE = \frac{1}{2}\varrho\varphi\frac{\partial\varphi}{\partial n}dS$$

or

$$E = \frac{\varrho}{2}\int\varphi\frac{\partial\varphi}{\partial n}dS \tag{9.1}$$

[Compare Division A IX (2 3)]

It should be noted that this is a surface integral and must be taken completely over the surface of the generating body in contact with the fluid. In the case of a thin circular disk moving in a two-dimensional field, for example, it must be taken all the way around the circumference of the disk. In the case of a straight line (thin rod) moving in a two-dimensional field, it must be taken on both sides of the line.

10. Virtual Mass. We have seen in 8 that the establishment of a field of flow requires the expenditure of a certain amount of work which is again represented in the field by the distributed energy of motion. The condition, so far as work and energy are concerned, is the same as though the impulse should act upon a certain mass M, producing a certain velocity V (that of the generating body) and an energy $M\,V^2/2$. The results, therefore, are the same as though there were a certain fictitious or *virtual mass* associated with the generating body, itself assumed to be without mass. This virtual mass M will therefore be found by equating the actual energy of the field, as in (9.1), to the energy of this virtual mass M with the velocity V. This gives

$$M = \frac{\varrho}{V^2}\int\varphi\frac{\partial\varphi}{\partial n}dS \tag{10.1}$$

11. Pressure at any Point in a Field Undergoing a Time Change. In all problems dealing with the conditions of steady motion and not with its establishment, all conditions are assumed to be constant in time. In particular in 2 the component velocities u, v, and w, were assumed to be constant in time and functions only of position in the field as defined by the space coordinates x, y, and z. Let us now take the more general problem and assume that all conditions vary in time as well as in space. This will be, in effect, a generalization of 5 to include variation in time as well as in space. Without repeating the somewhat detailed treatment of that section, we may, without any loss of generality assume an axis S in the direction of the motion at any point and thus, putting V for the total velocity, write

$$V = f(s, t) \text{ and}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial t}$$

or

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} \tag{11.1}$$

But with external forces absent (5.1) gives

$$\frac{dV}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial s}$$

and if the field has a velocity potential,

$$V = \frac{\partial \varphi}{\partial s} \quad \text{and} \quad \frac{\partial V}{\partial t} = \frac{\partial^2 \varphi}{\partial t \, \partial s}$$

Hence (11.1) becomes $-\frac{1}{\varrho}\frac{\partial p}{\partial s} = \frac{V \partial V}{\partial s} + \frac{\partial^2 \varphi}{\partial t \, \partial s}$

Integrating relative to s, we then have

$$-\frac{p}{\varrho} = \frac{V^2}{2} + \frac{\partial \varphi}{\partial t}$$

or

$$-p = \frac{\varrho V^2}{2} + \varrho \frac{\partial \varphi}{\partial t} \tag{11.2}$$

But from (8.3) we may substitute for potential in terms of impulse and thus write,

$$-p = \frac{\varrho V^2}{2} + \frac{\partial \omega}{\partial t}$$

This equation is seen to be a generalization of (6.4) and between any two points 1 and 2 we shall have

$$-p\Big]_1^2 = \frac{\varrho V^2}{2}\Big]_1^2 + \frac{\partial \omega}{\partial t}\Big]_2^1 \tag{11.3}$$

This may be considered as a generalized form of the Bernoulli equation. In particular it is seen that in the general case, the change in pressure between two points 1 and 2 is made up of two parts, one the change in the dynamic pressure $\varrho V^2/2$, as in the Bernoulli equation, and the other, the difference between the two time rates of change of the impulse ω at these two points.

CHAPTER II

PLANE IRROTATIONAL FLOW

1. Two-Dimensional Flow. We have thus far developed the basic equations for fluid motion assuming the general case of a three-dimensional field. There is, however, a numerous class of problems, especially in the applications of aerodynamics to aeronautics, for which a two-dimensional space will serve. The two-dimensional flow of a fluid may be pictured as the flow taking place in a very thin stratum or layer between two indefinite boundary planes. The layer may be conceived of as thin to a vanishing degree—at the limit, of the thickness of one molecule of the fluid.

On the other hand, we may conceive of a large number of such layers piled one on another, the motion in all layers the same, and thus build up the concept of a space of three dimensions but in which the phenomena of fluid flow are all accounted for by a study of the flow

in any one of these layers. Thus suppose we have a cylinder of very great length past which there is flowing, in a direction ⊥ to the length of the cylinder, a fluid medium of indefinite extent. Then the lines of flow past the cylinder, for all parts of the cylinder far from the ends, may be assumed to take place in a series of planes ⊥ to the axis of the cylinder, the motion in all of these planes being the same, since the conditions of flow in all are the same. We may at the limit consider the medium infinite in extent and the cylinder infinite in length. Then for any finite length of the cylinder, the lines of flow will be as above and a complete account of such motion will be secured through an examination of the motion in any one of these planes ⊥ to the axis of the cylinder. The entire space field of motion for any finite length of the cylinder is thus made up of a series of entirely similar patterns of motion in parallel thin layers. Such motion is then properly called two-dimensional, although the field as a whole may occupy a finite three-dimensional space.

The concept of a body of uniform cross section and of infinite length past which is moving, in a direction ⊥ to its length, a fluid medium of infinite extent, is thus a useful device for picturing the conditions in a finite part of space where, due to the constancy of all circumstances affecting the flow, it must occur in a series of similar patterns, in planes ⊥ to the length of the body, and thus constitute mathematically, a condition of two-dimensional flow. Flow of this character may thus be referred either to a two or a three-dimensional space. We may speak of the flow in space past an infinite cylinder or of the flow in a plane past a circle. In the present chapter we shall usually employ the latter form, but it should always be understood that whatever is established for the flow in a plane will hold likewise for an infinite summation of such planes making up an infinite field in three-dimensional space.

Taking then the plane of flow as that of $X\,Y$, it is usual to take the axis of X parallel to the direction of the relative motion at a point far from the solid bodies or boundaries with which we are concerned. The axis of Y is then taken ⊥ to X at any convenient location in the field under observation.

The notation for the treatment of three-dimensional flow in I 2 becomes immediately available for two-dimensional flow by the omission of the axis of Z and of the velocity component w.

For many purposes, however, polar coordinates are found more convenient than rectangular. In such case, we shall employ notation as follows:

$r =$ the radius vector.
$\theta =$ the angle made by r with the reference line (usually the axis of X).
$n =$ the component velocity in the direction of r.
$c =$ the component velocity in the direction ⊥ to r.

For two-dimensional flow the equation of continuity I 4 becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.1}$$

Equations I (5.3) become reduced to:

$$\left.\begin{aligned} -\frac{1}{\varrho}\frac{\partial p}{\partial x} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ -\frac{1}{\varrho}\frac{\partial p}{\partial y} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{aligned}\right\} \tag{1.2}$$

while the Bernoulli equation I (6.5) . . . (6.9), remains the same in form.

2. Rotational and Irrotational Motion. The condition and limitations under which the basic equations of fluid flow have been developed have already been set forth in I 2 . . . 6. Under these conditions it appears that all fluid flow (incompressible fluids) must meet the two requirements expressed by the equation of continuity and by Bernoulli's equation, which is, in effect, a statement of the conservation of energy. We may next subdivide this broad field of possible fluid movement into two types or forms, according as the movement of a small particle or element of the fluid consists of a motion of translation only, or of translation combined with rotation.

The general subject of *rotational* and *irrotational motion* has been considered in Division A and to which reference may be made. Referring to VI 8 of that division, it will be seen that the conditions for irrotational motion are precisely those for the existence of a potential for a vector V with X and Y components u and v (see Division A VII 4). In other words, if the motion is irrotational, there will exist a velocity potential and if this is denoted by φ, we may write,

$$\left.\begin{aligned} u &= \frac{d\varphi}{dx} \\ v &= \frac{d\varphi}{dy} \end{aligned}\right\} \tag{2.1}$$

where u and v mean specifically the X and Y component velocities in the field of flow. More broadly, and following the basic definition of a vector potential, as in Division A VII 1, it follows that the resultant velocity in any direction in the field of flow will be measured by the derivative of the function φ taken in this particular direction.

It results that the function φ, if known, is thus able to give us a complete account of the field of flow through the component velocities u and v determined as above, or more broadly through any combination of component velocities taken in such directions as may be convenient.

Thus if φ is expressed in polar coordinates, we find immediately the components along and $\perp$ to the radius vector, as in Division A VII 5

$$\left.\begin{aligned} n &= \frac{d\varphi}{dr} \\ c &= \frac{d\varphi}{r\,d\theta} \end{aligned}\right\} \tag{2.2}$$

The function φ, expressed as a function of x and y for example, thus enfolds within itself as a simple algebraic expression, the value of an infinite number of velocities, and all or any to be had for the simple price of a partial differentiation along the direction in which the velocity is desired and of substitution of the coordinates of the particular point in question. Or more practically we have only to find by partial differentiation the u and v components at the point in question and then the resultant V by combination of these.

It is this property which gives to a velocity potential its great value in the investigation of problems of fluid flow, as will appear in detail in subsequent pages.

But in Division A VII **5** it has been shown that the functions φ and ψ arising out of the development of a function of $(x + iy)$ meet, not only the requirement for irrotational motion, but likewise the condition $\nabla^2\varphi = 0$ or $\nabla^2\psi = 0$, for flow as an *incompressible medium.* These functions are, therefore, ready to hand for use as *velocity potential* and *stream functions* for all cases of flow in two dimensions. Putting the complete function in the form

$$w = f(x + iy) = \varphi + i\psi \tag{2.3}$$

the relations of φ and ψ to a field of fluid flow are there shown in detail and need not be here repeated. It is clear, however, that for any field of fluid flow for which we know or can find the functions φ or ψ, we have at hand immediately, the means for a complete knowledge of the field, geometrically and dynamically, and to any degree of detail which may be desired.

3. Fields of Flow. It will be desirable at this point, to examine the characteristics of a number of different modes of irrotational motion in two dimensions, and to derive the velocity and stream functions φ and ψ and their combination in the general potential function w.

For convenience we repeat at this point the basic relations between these functions and the component velocities u, v, n, c.

$$\left.\begin{aligned}
u &= \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} & d\varphi &= u\,dx + v\,dy \\
v &= \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} & d\varphi &= n\,dr + c\,r\,d\theta \\
n &= \frac{\partial\varphi}{\partial r} = \frac{\partial\psi}{r\,\partial\theta} & d\psi &= -v\,dx + u\,dy \\
c &= \frac{\partial\varphi}{r\,\partial\theta} = -\frac{\partial\psi}{\partial r} & d\psi &= n\,r\,d\theta - c\,dr \\
& & w &= f(\,.\,) = \varphi + i\psi \\
& & \frac{dw}{dz} &= u - iv
\end{aligned}\right\} \tag{3.1}$$

See Division A VII (5.2), (5.3), (5.4), (5.5), (5.6).

4. Rectilinear Flow Parallel to Axis of X. For such a case we have:

$$u = U \text{ (constant)}$$
$$v = 0$$

Then:
$$\frac{\partial \varphi}{\partial x} = U \qquad \frac{\partial \varphi}{\partial y} = 0$$

and
$$\varphi = U x \tag{4.1}$$
$$\frac{\partial \psi}{\partial y} = U \qquad \frac{d \psi}{d x} = 0$$

and
$$\psi = U y \tag{4.2}$$
$$w = \varphi + i\psi = U\,(x + i y) = U z \tag{4.3}$$

The stream-lines are all straight lines parallel to X, lying above X for ψ positive and below X if ψ is taken as negative.

5. Rectilinear Flow Parallel to Axis of Y. Let

$$u = 0$$
$$v = V \text{ (constant)}$$

Then:
$$\frac{\partial \varphi}{\partial y} = V \qquad \frac{\partial \varphi}{\partial x} = 0$$

and
$$\varphi = V y \tag{5.1}$$
$$-\frac{\partial \psi}{\partial x} = V \qquad \frac{\partial \psi}{\partial y} = 0$$
$$\psi = - V x \tag{5.2}$$
$$w = \varphi + i\psi = V\,(y - i x) = - i V z \tag{5.3}$$

The stream-lines in this case are straight lines parallel to Y, lying on the left of Y for ψ positive and on the right for ψ negative.

6. Rectilinear Flow Oblique to Axes. Let:

$$u = U \text{ (constant)}$$
$$v = V \text{ (constant)}$$

Then
$$d\varphi = \frac{\partial \varphi}{\partial x} d x + \frac{\partial \varphi}{\partial y} d y = U\,d x + V\,d y$$

and
$$\varphi = U x + V y \tag{6.1}$$
$$d\psi = \frac{\partial \psi}{\partial x} d x + \frac{\partial \psi}{\partial y} d y = - V d x + U d y$$
$$\psi = - V x + U y \tag{6.2}$$
$$w = \varphi + i\psi = U x + V y - i\,(V x - U y)$$
$$= U\,(x + i y) - V\,(i x - y)$$
$$w = U z - i V z = (U - i V) z \tag{6.3}$$

The latter result comes directly from (3.1). Thus

$$\frac{d w}{d z} = (u - i v) = (U - i V)$$
$$w = (U - i V) z, \text{ as above.}$$

The stream-lines in this case are straight lines inclined to X at an angle whose tangent is V/U. Zero reference line for ψ is given by $Uy = Vz$ [see (6.2)] or $y = (V/U)x$ which is a line through O inclined to X at the angle $\tan^{-1} V/U$.

7. Sources and Sinks. In the study of problems in fluid mechanics, much use is made of the concept of *Sources* and *Sinks* We shall first assume a two-dimensional flow and later generalize certain of the results to the case of a three-dimensional flow.

In a two-dimensional flow, a source, as at O Fig. 5, is conceived of as a point from which fluid issues, spreading over the plane surrounding O, equally in all directions. It results from this picture that the lines of flow will be straight radial lines. Let r_1 and r_2 denote the radii OA and OB and n_1 and n_2 the corresponding radial velocities across the arcs AD and BC. Then if θ is the angle AOD, we shall have:

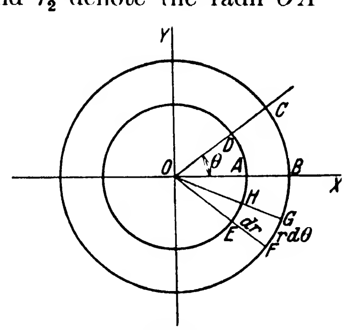

Fig. 5.

$$\text{Flow across } AD = r_1 n_1 \theta$$

$$\text{Flow across } BC = r_2 n_2 \theta$$

But from the condition of continuity, we must have

$$r_1 n_1 \theta = r_2 n_2 \theta$$

or

$$\frac{n_1}{n_2} = \frac{r_2}{r_1} \tag{7.1}$$

Hence the velocity will vary inversely as the radius or distance from the source.

From (7.1) we have:

$$r_1 n_1 = r_2 n_2 = rn = \text{constant} \tag{7.2}$$

Whence, $2\pi rn = $ constant.

But $2\pi rn$ is the total flow across a circular boundary of radius r, and (7.2) states that this must be a constant, as indeed is obvious. This constant is termed the *strength* of the source.

This we shall denote by m. We have then

$$2\pi rn = m \quad \text{or} \quad n = m/2\pi r \tag{7.3}$$

as the general equation of flow for a source.

A *sink* is simply the reverse of a *source*. It is a point into which fluid flows equally from all directions in the environment. Algebraically a sink is represented as the negative of a source.

8. Functions φ and ψ for Source. We have now to inquire as to whether for the distribution of velocity in the field of flow for a source, there can exist a velocity potential, and if so, as to its form. Following the previous strategy we have simply to express the component velocities u and v as functions of x and y and then seek to integrate as in the previous cases.

Now at any point in the field, as C Fig. 5, the total velocity $n = m/2\pi r$ is directed along the radius outward. Or writing $\mu = m/2\pi$ we shall have for the two components of n

$$u = \frac{\mu}{r}\cos\theta = \frac{\mu x}{x^2 + y^2}$$

$$v = \frac{\mu}{r}\sin\theta = \frac{\mu y}{x^2 + y^2}$$

Then: [see (3.1)]
$$d\varphi = \frac{\mu x dx}{x^2 + y^2} + \frac{\mu y dy}{x^2 + y^2} \tag{8.1}$$

Remembering that the first of these is a partial derivative with reference to x and the second with reference to y, we must integrate accordingly, that is, the first as though y were constant and the second as though x were constant.

The first integration gives

$$\frac{1}{2}\mu \log(x^2 + y^2)$$

and the second the same. Inversely, it is seen that the partial derivative of this expression with regard to x, will give the first term on the right in (8.1), and with reference to y, the second term. Thus we have:

$$\varphi = \frac{1}{2}\mu \log(x^2 + y^2)$$

Or putting
$$r^2 = (x^2 + y^2)$$

$$\varphi = \mu \log r \tag{8.2}$$

For φ constant, the value of r is a constant and hence the curves of constant φ are circles about O as center.

Next to determine the stream function ψ we have, as in (3.1)

$$d\psi = \frac{\mu x dy}{x^2 + y^2} - \frac{\mu y dx}{x^2 + y^2}$$

Integrating the first with respect to y gives

$$\mu \tan^{-1}\frac{y}{x}$$

and the second with respect to x, the same result. This again may be checked by taking the derivative first with respect to y and second with respect to x. We have therefore:

$$\psi = \mu \tan^{-1}\frac{y}{x} = \mu\theta \tag{8.3}$$

This result might, indeed have been derived directly from the definition of ψ as the total flow between some one of the lines $\psi =$ *constant*, taken as datum, and any other line in general. If then we take the axis of X as the line of $\psi = 0$, we shall obviously have, between X and any stream-line at an angle θ, a total flow $\mu\theta$ and hence in general,

$\psi = \mu\theta$ as above.

We have, thus, a complete description and characterization of this field of flow in terms of the functions φ and ψ as above.

The complete potential function w follows from φ and $i\psi$ [see Division A I **14** (b)] and in the present case we have,

$$w = \mu \log (z) = \mu \log (x + i y) \tag{8.4}$$

The preceding development has been by way of rectangular coordinates and as an illustration of the procedure in the use of this system. The same results are reached much more directly through polar coordinates as will now be shown.

We have: $$\frac{\partial \varphi}{\partial r} = n = \frac{\mu}{r} \qquad \frac{\partial \varphi}{r\,\partial \theta} = 0$$

Hence $$\varphi = \mu \log r \tag{8.5}$$

$$\frac{\partial \psi}{r\,\partial \theta} = n = \frac{\mu}{r} \qquad \frac{\partial \psi}{\partial r} = 0$$

Hence $$\frac{\partial \psi}{\partial \theta} = \mu \qquad \text{and} \qquad \psi = \mu\,\theta \tag{8.6}$$

From Division A VII 5 we know that if we have φ and ψ as the expansion of $f(x + i y)$, then the velocities u and v, representing partial derivatives of φ or ψ, will fulfil the conditions for irrotational flow. It is, however, of interest to show directly that this condition is fulfilled by the motion as specified. Referring to Fig. 5, we remember that with irrotational flow, the line integral of the velocity around any small element of the field will be zero. Let $EFGH$ be any such element with sides dr and $rd\theta$. The velocity along EH and FG is 0. The velocity along EF equals that along HG. Hence in going around the area $EFGHE$, the line integral along $EF = n dr$, that along FG is zero, that along $GH = - n dr$ and that along HE is zero and the sum is zero. Hence the motion is irrotational.

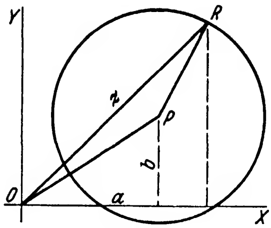

Fig. 6.

Equations for Source with Origin Not at Source Point. For use in subsequent problems, it is of interest to derive here the form of the functions φ and ψ referred to an origin not at the source point. In Fig. 6, P is the location of the source and O the origin of coordinates. Then referring to (8.2) it is clear that for φ, we have simply to express the distance PR in terms of the coordinates x and y of the point R.

Hence, $$\varphi = \frac{\mu}{2} \log [(x - a)^2 + (y - b)^2] \tag{8.7}$$

Likewise from (8.3) $$\psi = \mu \tan^{-1} \frac{(y - b)}{(x - a)} \tag{8.8}$$

For w, we need, as in (8.4), simply the vector expression for the distance PR referred to O as origin. As vectors we then have:

$$
\begin{aligned}
z &= OP + PR \\
PR &= z - OP \\
OP &= a + b \\
PR &= z - (a + b)
\end{aligned}
$$

Hence, $$w = \mu \log PR = \mu \log [z - (a + b)] \tag{8.9}$$

If P is located on X or Y, b or a becomes 0, and the expressions are simplified to that extent.

Again if P is located in the second third or fourth quadrant, the signs of a and b will be modified accordingly.

9. Functions φ and ψ for Sink. A sink, as noted above, is the exact reversal of a source. If the lines of flow of a source are reversed, we shall then have the field of flow for a sink.

It is readily seen that for the function ψ, we shall have the same as for a source but with the sign reversed with the reversal of the direction of flow, $$\psi = -\mu\theta \tag{9.1}$$
while for the function φ we shall have the same in value again with the sign reversed: $$\varphi = -\mu \log r \tag{9.2}$$
and for the potential function $$w = -\mu \log (z) \tag{9.3}$$

A source may, in effect, be considered as having a strength $+m$ and a sink as having a strength $-m$ and any equation or expression involving the one may be transformed to the other by a change in the sign of $\mu - m/2\pi$.

CHAPTER III
VORTEX FLOW

1. Vortex Flow. The fields of flow thus far considered fall under two general types:

(1) Indefinite rectilinear flow.

(2) Flow resulting from a source or a sink.

There is a third important type of flow to which we must now turn our attention—that of *vortex flow* or *vortex motion.*

We must first define a vortex flow in two dimensions. In Fig. 7 let O be the center of a thin plane sheet of rotating fluid. It is important to note here that this fluid is not assumed to rotate as a rigid sheet. As a fluid without viscosity, the successive circular filaments may have any velocity in themselves or any relative velocity of sliding without resistance or loss of energy in the field.

We have then to inquire as to the radial distribution of velocity which will give irrotational motion, and thus insure a potential for the field. We shall find it convenient, however, to exclude from consideration at the center, a circle of radius a and to consider only the field of flow lying outside of this circle. This inner limit radius a may have any

value as desired. The field within this circle may then be conceived of as without fluid, the limit circle forming a barrier around which the vortex field revolves, or it may be conceived of as forming a second rotating field under some other law, for example, as a rigid mass.

For the moment then, we are only concerned with the field outside this circle of radius a and over this field we wish to develop the conditions for irrotational motion. It may at first sight appear inconsistent to assume that an element of the field at P, Fig. 7, can move in a circular path about O as a center and yet meet the conditions of irrotational motion. It will be remembered, however, that these conditions require simply that the element itself shall not rotate and for which the mathematical condition may be expressed in either of two ways:

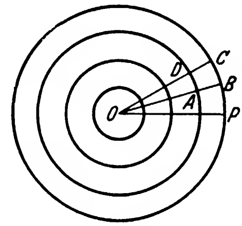

Fig. 7.

(a) $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.

(b) The *circulation, i. e.*, the line integral of the velocity around any small element of the field = zero. And obviously if this condition is fulfilled for any and all small elements in the field, it will be fulfilled for any aggregate of such elements and for any closed contour whatever in the field.

From the condition in form (b), we easily derive the law of the velocity in the circular paths. In Fig. 7 let $ABCD$ be a closed path made up as indicated. The velocity along the two radial lines AB and CD will be zero and will, therefore, contribute nothing to the line integral. Let r_1 and r_2 be the two radii OA and OB and c_1, c_2 the two velocities along AD and BC. Then in going around the area in the order $ABCDA$ we shall have $r_2 c_2$ positive and $r_1 c_1$ negative and hence condition (b) will become $r_2 c_2 - r_1 c_1 = 0$

or
$$\frac{c_2}{c_1} = \frac{r_1}{r_2} \tag{1.1}$$

or
$$c \sim \frac{1}{r}$$

That is, the velocity must vary inversely as the distance from the center.

Again from (1.1) we have

$$2\pi r_1 c_1 = 2\pi r_2 c_2$$

That is, the line integral of the velocity, or the circulation, around any and all circular paths with O as center will be constant. This constant value is called the ***strength of the vortex*** or the ***vorticity*** and will be denoted by Γ, or sometimes γ.

Whence, for any circular path:

$$\text{Circulation} = \Gamma = 2\pi r c \tag{1.2}$$

For an arc $d\theta$ of such a path, we shall have,

$$d\,(circ) = c\,r\,d\,\theta = \frac{\Gamma}{2\,\pi}\,d\,\theta \tag{1.3}$$

We must next generalize this statement regarding the circulation around a circular path, to include any and all closed paths which inclose the vortex center.

In Fig. 8 let PQR represent any closed path whatever inclosing the vortex center O. Let OA, OB, etc. be a series of radii spaced with a constant small angular interval $d\theta$. Then for the actual path, we may substitute the broken line path $ABCD$ etc., made up of portions of radii and arcs of circles. At the limit such a broken line path will represent the actual path within a difference less than any assignable quantity. But the velocity along the radial lines will be zero and the velocity along any circular arc will vary inversely as the radius while the length of the arc will vary directly as the radius. Hence the product, or the elements of the circulations along such arc will be constant and independent of the radius, and measured as in (1.3), by the value

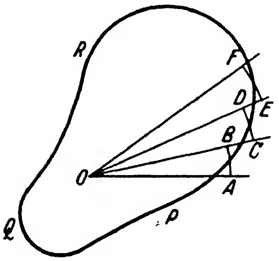

Fig. 8.

$$d\,(circ) = \frac{\Gamma}{2\pi}\,d\,\theta$$

But for the complete path PQR the summation of $d\theta$ will be $2\,\pi$ and for such path, $\quad circ = \frac{\Gamma}{2\,\pi}\,2\,\pi = \Gamma = a$ constant

Hence for any and all paths containing O within the path, the circulation will be the same and equal to the vortex strength.

This brings us to an important qualification regarding the conditions for irrotational motion in a vortex field. As stated in (a) or (b) the condition requires that the circulation around any closed path in the field shall be zero. We now see that this statement must be understood as applying only to closed paths which *do not* include the vortex center. We may, therefore, state the general conditions more fully as follows:

In a vortex flow as defined, the circulation around any and all closed paths which include the vortex center will be constant and equal to Γ, the vortex strength or vorticity. The circulation around all closed paths which do not include the vortex center will be *zero.*

It will be clear that the velocity relations upon which these statements depend will only hold in a field without obstructions to the flow, such as would be represented by solid obstacles interrupting the continuity of the field. Reference here may be made to Division A VI 4 regarding reconcileable paths and singly and multiply connected regions. It thus follows that the statement that the value of the circulation about a

closed path is independent of the shape of the path must be understood as referring to paths which are reconcileable one with another, i. e. to paths such that one may be deformed continuously into another without being cut by a solid obstruction, or by a vortex center.

A vortex flow such as here assumed could not exist by itself in an indefinite body of fluid. If, however, we assume an indefinite series of such vortex flows with their centers or cores distributed along a line and with the plane of each flow ⊥ to the tangent to the line at its center, we shall have the picture of a space vortex field.

The summation of the centers of the vortex flows will thus form a continuous line known as a vortex filament, and about this filament or core the vortex movement will be distributed always in planes ⊥ to the core at the point where the latter pierces the plane.

From these properties of a field containing a vortex line, it is readily shown that the vortex strength of such a line must be constant throughout its length. This may be seen as follows: Let PQ Fig. 9 be any indefinite vortex line. Then take a path $ABCDEFA$ such that the lines AB and ED are very near while BCD and EFA are two circuits about the line except for the small gaps DB and AE. The circulation along this path as a whole must be zero since it does not include the vortex line. The circulations along AB and DE, equal and opposite in direction, will mutually cancel. Hence the circulations about BCD and EFA must also mutually cancel. But these latter are the circulations about the vortex line (except for the infinitesmal gaps) taken in opposite directions at two different points along the line. Hence, since they must cancel they must be numerically equal, or in other words the circulation about the vortex line taken in the same cyclical direction will be constant at all points along the line.

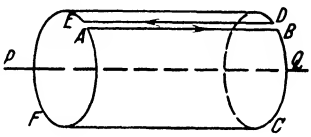

Fig. 9.

In like manner as with stream-lines in Division A VI 5, we may have an assemblage of vortex lines or filaments touching the points on a closed curve and forming thus a vortex tube. Again this tube may be conceived of as filled with vortex lines in the same manner as a tube of flow is filled with stream-lines; and since the strength of a vortex line cannot change along its length, the circulation around such a vortex tube will be constant throughout its length in the same manner as the flow along a tube of stream flow.

For present purposes, the following additional brief statements of certain results of the theory of vortex flow as developed by Helmholtz, Kelvin and others, may be here given. Some further developments of vortex theory will be found in Division C.

(1) The vortex core may be straight or curved.

(2) In an indefinite fluid a vortex core cannot end in the fluid. It must either end on a solid surface or form a closed circuit. The common smoke ring is an example of a vortex with a circular core.

(3) Kelvin's theorem applied to any closed curve which encloses a vortex line leads to the conclusion that all the fluid particles which are part of a vortex motion at any time, remain a part of it for all time.

(4) In a perfect fluid, as assumed, there is no conceivable way in which a vortex could be generated by external agency; and if once in existence there is no conceivable way in which a vortex could be destroyed or annulled.

(5) Stokes theorem applied to the circulation around a circuit enclosing a number of vortex lines or filaments shows that the circulation around the circuit must be equal to the sum of the circulations about the individual vortex lines, that is, to the algebraic sum of the strengths of all the vortex lines which cut the surface bounded by this circuit.

It follows that if we imagine such a closed circuit moved in a field of vortex lines, the circulation around such circuit will increase or decrease in accordance with the change in the vortex lines which cut this surface.

2. Induced Velocity. An important application of the concept of a vortex and of vortex motion arises in connection with computations of the velocity in a field surrounding a system of vortex cores. To approach this problem we must first extend somewhat the definition given in the preceding section to Γ as the vorticity or vortex strength. As there defined, $\Gamma = 2\,\pi\,r\,c$ has purely a geometric or kinematic quality, confined to the plane which represents the vortex flow in question. With a line vortex, however, we must introduce the concept of a third dimension—that directed along the line of the vortex core. To this end we now define $\Gamma = 2\,\pi\,r\,c$ as the strength of the vortex per *unit length* of core, and following this definition we may say that the strength of the vortex for a core length $d\,x$, or x, or a, will be $\Gamma\,d\,x$, Γx, or Γa.

In Fig. 10a let $X\,X$ be an indefinite straight line vortex core. Then there will exist in the space around $X\,X$ a field of vortex motion in planes $\perp$ to $X\,X$ and with a distribution of velocity outward from $X\,X$ as we have seen. Now in certain aeronautic problems, it becomes desirable to ascertain the velocity field which would be consistent with the existence of a part only of this vortex core. As we have seen, a vortex must either end on solid surfaces or form a closed circuit, or as a third possibility, it may be assumed to extend indefinitely or to infinity. Thus the vortex core $X\,X$ extending indefinitely in either direction will meet the requirements for the stability of a vortex system. Again instead of extending in a straight line to ∞, as in Fig. 10a, it is possible to conceive of a vortex core in a U form, see Fig. 10b, extending to ∞ at $X\,X$, and connected across at $A\,B$ with a branch at right angles to $A\,X$ and $B\,X$.

Without developing further these details, it will be sufficient to note that in many problems in aeronautics we may wish to develop the field distribution of velocity conceived of as due to a part AX or AB of this complete vortex system. To this end it is necessary to develop an expression for the value of the field conceived of as consistent with or due to a small or differential element of the vortex core. The problem thus reduces itself to the following:

Given an indefinite vortex core XX, Fig. 10*a*. Consider any point P in its relation to an element of length dx at a point Q. Then we wish to find a measure of the velocity at P consistent with the existence of an element dx at Q of a vortex core of strength Γ.

The velocity at P is commonly spoken of as the *induced velocity* or the velocity at P induced by the action of the vortex core element at Q. This language is not altogether fortunate, since it is not easy to picture a causal relation between the core and the velocity at P. Rather they are local values in a field distribution of vortex motion. This particular picture of the relation between the field velocity and the vortex core arises naturally, however, having in mind the parallelism between the vortex problem and the problem of the strength of the magnetic field in the space surrounding a conductor carrying electric current. The strength of the magnetic field corresponds to the field velocity, and the current strength to the vortex strength. The analogy is complete and the same laws of distribution hold in both cases. Since then, the magnetic field is conceived of as induced or called into being by the action of the current, so it has become customary to speak of the velocity in the vortex field as induced by the vortex core. While it would presumably be more correct to say, as above, that the field velocity is consistent with, or correlative to, the existence of the vortex core, we shall, in what follows, employ the usual term and speak of this as the induced velocity.

Furthermore, while this induced velocity is of the same geometrical character as the general vortex velocity denoted above by c, it is found convenient to denote induced velocity by a separate character and for this we shall employ w.

With this comment regarding nomenclature and notation we are now ready to proceed with the development of an expression which will represent the velocity induced at P conceived of as due to the action of the element dx at Q of a vortex core of strength Γ.

What we seek is really a mathematical expression for the *assumed* action of an element dx, which when integrated for an indefinite line, as from $-\infty$ to $+\infty$, will give the result which is required by the conditions of a vortex field. More specifically, in Fig. 10*a*, for the circulation around the circle passing through P and with a as radius,

Γ being the strength of the vortex core, we know that these conditions call for a circulation measured by

$$2\pi a w = \Gamma$$

Whence $$w = \frac{\Gamma}{2\pi a}$$

The problem is, then, to derive an expression for the action at a point P of an element dx at any point Q, which, when integrated between $-\infty$ and $+\infty$ will give the value $w = \Gamma/2\pi a$. To the problem put in this way there is no direct solution. Fortunately, however, the electrical analogy does furnish a more definite physical basis for the expression of the elementary force dF acting between a unit magnet pole at P and an element dx of current strength i at a point Q. This is in the form

$$dF = \frac{A\, i\, dx \sin\theta}{r^2} \tag{2.1}$$

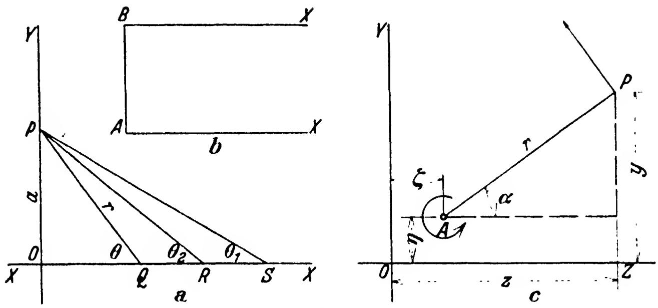

Fig. 10.

where A is a constant depending on the units employed. Whence

$$F = A\, i \int \frac{dx \sin\theta}{r^2}$$

Putting the integral all in terms of θ we have:

$$x = -a \cot\theta$$
$$dx = a \operatorname{cosec}^2\theta\, d\theta$$
$$r = a \operatorname{cosec}\theta$$

and $$F = \frac{A\, i}{a} \int_0^{\pi} \sin\theta\, d\theta = -\frac{A\, i}{a} \cos\theta \Big]_0^{\iota} \tag{2.2}$$

Noting, as in Fig. 10a that in passing from $x = +\infty$ to $-\infty$, θ must always be measured on the same side of the line PQ, we shall have for the corresponding limits for θ, 0 and π. Substituting these limits gives

$$F = \frac{2\, A\, i}{a}$$

In the problem of the vortex motion of a fluid F is the analogue of the velocity w and i of the vortex strength Γ. Hence for the vortex problem we may write: $w = \frac{2 A \Gamma}{a}$

But to fulfil the conditions of the vortex field we know that

$$w = \frac{\Gamma}{2 \pi a}$$

Whence, $A = \frac{1}{4 \pi}$

For the element of the induced velocity corresponding to an element $d x$ of a vortex core, we may then write (2.1) as follows:

$$d w = \frac{\Gamma d x \sin \theta}{4 \pi r^2}$$

and this integrated will give parallel with (2.2),

$$w = \frac{\Gamma \cos \theta}{4 \pi a} \Bigg|_{\theta_2}^{\theta_1}$$

Thus for any piece of the vortex core RS we shall have at P

$$w = \frac{\Gamma (\cos \theta_1 - \cos \theta_2)}{4 \pi a}$$

For the indefinite length from $x = 0$ to infinity in either direction this becomes $w = \frac{\Gamma}{4 \pi a}$

and the sum of these values for both directions gives for an infinite rectilinear vortex core $w = \frac{\Gamma}{2 \pi a}$

which brings us back to the value with which we started.

The electro-magnetic relation upon which this development has been based is commonly known as the law of Biot and Savart, and in application to the problems of vortex motion, reference is usually made to it under this name.

A further application of these results, of frequent and important application in aeronautic problems is illustrated in Fig. 10c which may be considered as giving at A an end view of the vortex line $X X$. Relative to the origin O the coordinates of this point are ζ, η, while those of the point P are z, y.

Assuming the vortex line of indefinite length and of strength Γ, the induced velocity at P in a direction $\perp$ to $A P$ will be

$$w = \frac{\Gamma}{2 \pi r}$$

This will have components along Z and Y as follows:

$$w_z = \frac{\Gamma}{2 \pi r} \sin \alpha$$

$$w_y = \frac{\Gamma}{2 \pi r} \cos \alpha$$

Expressed in terms of the coordinates these become,

$$\left.\begin{aligned} w_z &= \frac{\Gamma(y-\eta)}{2\pi\,[(z-\zeta)^2+(y-\eta)^2]} \\ w_y &= \frac{\Gamma(z-\zeta)}{2\pi\,[(z-\zeta)^2+(y-\eta)^2]} \end{aligned}\right\} \qquad (2.3)$$

These same principles and methods are readily extended to problems in three dimensions.

3. Functions φ and ψ for Plane Vortex Flow. We have seen that a field of vortex flow defined in such manner as to meet the requirements for a potential function has component velocities along and $\perp$ to the radius as follows:

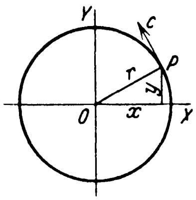

Fig. 11.

$$n = \text{velocity along radius} = 0$$

$$c = \text{velocity} \perp \text{to radius} = \frac{\Gamma}{2\pi r}$$

Putting $k = \Gamma/2\pi$ this becomes

$$c = \frac{k}{r}$$

Let P Fig. 11 denote any point in such a field. Then from II (3.1)

$$d\varphi = c\,r\,d\theta + n\,d\,r$$

But $n = 0$ and substituting for c we have

$$d\varphi = \frac{k}{r}\,r\,d\,\theta = k\,d\,\theta$$

or $$\varphi = k\theta \qquad (3.1)$$

Again $$d\psi = n\,r\,d\theta - c\,d\,r$$

But again $n = 0$ and substituting for c we have

$$d\psi = -\frac{k}{r}\,d\,r$$

or $$\psi = -k\log r \qquad (3.2)$$

Since we have excluded that part of the field lying within a circle of radius a, we must put limits of a and r in (3.2) and evaluate between.

This gives: $$\psi = -k\log\frac{r}{a}$$

As we have seen, there are no general conditions which limit the value of a. For the ideal vortex sheet, a may be taken very small or in other cases it may represent a circular barrier about which the vortex motion prevails.

The purpose of a definite inner limit a is in order to avoid the appearance of $\log 0$ in the value of ψ.

Combining (3.1) and (3.3) we have for the potential function,

$$w = k\,\theta - i\,k\log\frac{r}{a} \qquad (3.4)$$

For the expression of w as a function of z we proceed most conveniently with the rectangular component velocities u and v.

Thus $\quad u = -c \sin \theta \qquad v = c \cos \theta$

or
$$u = -\frac{k}{r} \sin \theta = -k \frac{y}{r^2}$$
$$v = \frac{k}{r} \cos \theta = k \frac{x}{r^2}$$

Then
$$\frac{dw}{dz} = u - i v = -k \left(\frac{y + i x}{r^2} \right) = -\frac{i k}{x + i y} = \frac{-i k}{z}$$

whence
$$w = -i k \log z \tag{3.5}$$

By reference to Division A I 14 (b) this becomes
$$w = k \left(\tan^{-1} \frac{y}{x} - i \log r \Big]_a^r \right) \tag{3.4}$$

which is the same as above.

Comparing the values of φ and ψ in (3.1) and (3.2) with the values for these functions for a source in II 8, it is seen that the function ψ for the source has the same form as the function φ for the vortex, while the function φ for the source has the same form as the function ψ for the vortex, with, however, the inverse sign. Many analogies and relations of interest, geometrical and physical may be developed from these mutual identities in form.

CHAPTER IV

COMBINATION FIELDS OF FLOW

1. Combinations of Fields of Flow. The addition theorem for vector potentials, in Division A VII 2 shows that for any combination of vector fields the resultant potential will be simply the summation of those for the individual fields. Hence the potential for any combination of fields of fluid flow will be given by the corresponding combination of the potentials of the fields individually. We may thus, starting with simple elements, build up, by a process of addition, combination fields of any degree of complexity. Before proceeding to a consideration of some of the more important of these combinations, we shall find it useful to consider certain general features found in combinations of this character.

Given two stream functions ψ_1 and ψ_2 with stream-lines at successive values of the functions, the intervals being equal and the same for both functions. Let the resulting system of lines be represented by Fig. 12. We have already seen that the difference between the two values of ψ for two stream-lines is a measure of the volume rate of flow in the stream between these lines. Such systems of lines as in Fig. 12 will therefore imply that the elementary stream flows between these successive pairs of lines are all equal and the same for each system, ψ_1 and ψ_2.

Let the directions of the flows be assumed as indicated by the arrows. In Division A VIII 2 it was shown:

(1) That the value of ψ at any point in a field of flow is measured by the total rate of flow between some stream-line taken as datum and the stream-line which passes through such point.

(2) That by the conventions of sign, flow lying on the left of the datum line, looking in the direction of the flow, must be counted as positive and that on the right as negative.

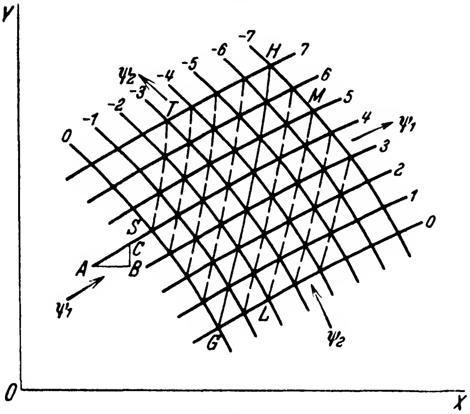

Fig. 12.

The direction of increase in the values of ψ_1 and ψ_2, Fig. 12, must, therefore, be: For the former, to the left and up, and for the latter, to the left and down. If then we choose arbitrarily for datum lines those marked zero, the successive stream-lines will be numbered as in the diagram (taking $\triangle \psi = 1$).

Now suppose these two fields combined. As we have seen, we shall have a new stream function $\psi = \psi_1 + \psi_2$. Certain geometrical results of such a combination will be of interest. The two sets of stream-lines will, as we have seen, divide the field into four sided elements of area as in Fig. 13. If now the unit of flow, that is the unit interval between successive values of ψ_1 and ψ_2, be taken very small, these elements, at the limit, will become small parallelograms as shown at $OPRQ$. Let V_1 denote the velocity at this element $OPRQ$ for ψ_1 and V_2 that for ψ_2. Then the flow for ψ_1 between OP and QR is $V_1 OQ \sin \theta$ and that for ψ_2 between OQ and PR is $V_2 OP \sin \theta$. But these are equal.

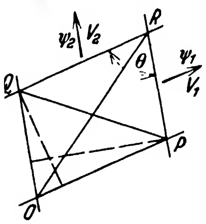

Fig. 13.

Hence $$V_1 OQ = V_2 OP$$

or $$\frac{V_1}{V_2} = \frac{OP}{OQ}$$

We may express this by saying that V_1 and V_2 are respectively proportional to OP and OQ or otherwise that the velocity along the stream-line of ψ_1 is proportional to the intercepts made by the stream-lines of ψ_2 and similarly for the velocity along ψ_2. It will then follow that the resultant velocity at O will be proportional to OR and in the direction OR. The stream-line at this point, resultant from the combination of these two modes of flow, will therefore lie along OR, the diagonal of the parallelogram.

If the elements of the field, instead of being indefinitely small, are finite in size with curved sides, as in Fig. 12, they may, of course, be considered as made up of a large number of the smaller elements, the summation of which will lead, by some form of path, across the diagonal from one corner to the other. It thus results, with two systems of flow, ψ_1 and ψ_2, with stream-lines laid off in both systems with the same interval $\triangle\psi$, that the resultant stream-lines will be indicated by tracing along the diagonals continuously from one element to another across the field, and as indicated in Fig. 12. The choice of the diagonal, as between OR and PQ, Fig. 13, must be made in accordance with the direction of the component velocities V_1 and V_2. Thus with these components in the direction OQ and OP, the diagonal must be OR. With either V_1 or V_2 reversed in direction, the diagonal would be PQ, or QP.

Now referring again to Fig. 12, it will be seen, with the numbering as developed out of the assumed directions of flow, that the sum of the numbers at the successive junctions along the resultant stream-lines is a constant for any one line. Thus along the line ST the sum is four; along LM it is negative two. This is, of course, only a result of the basic equation $\psi = \psi_1 + \psi_2 = \text{constant}.$

Thus ψ is the stream function for the combination, constant for any one resultant stream-line and with various numerical values as determined by its location in the field. Attention may be called to the line for $\psi = 0$, shown in Fig. 12 as GH.

It is readily seen that there is nothing unique with regard to this line. Its location is due wholly to the lines in the component fields ψ_1, and ψ_2 which were taken as datum or zero lines, and these are arbitrary. It results that GH is simply the line along which the algebraic sum of the component flows is zero. Such a line of zero flow in the new field, naturally becomes the new datum and the other lines number and take value from this in accordance with the algebraic value of $\psi = \psi_1 + \psi_2$.

At this point certain features of stream-line flow may be noted. In any field of permanent or steady motion flow the stream-lines retain their identity or individuality throughout the field. They do not cross. At special or singular points one line may touch another, but they do not cross and in general each line follows its course without contact with another. This again gives a picture of a small stream, such as that between two adjacent lines, with flow continuous and uninterrupted from one end of its course to the other. For illustration see any of the various diagrams of stream-line flow. The fluid thus moves in these small streams or ribbons of flow as though each stream-line were a smooth solid guide or barrier, marking out the path of flow. With this picture of the flow, it is clear that we can imagine any stream-line replaced by a thin smooth partition or barrier, without in any degree interfering with the flow on either side of such line. Again it will follow that the

total flow on either side of such barrier could be suppressed, without in any way interfering with the flow on the other side.

2. Two Rectilinear Fields, One Parallel to X and One Parallel to Y. Let U_1 and U_2 denote the two velocities parallel to X and Y respectively.

Then we have immediately:
$$\begin{aligned} \varphi &= U_1 x + U_2 y \\ \psi &= U_1 y - U_2 x \\ w &= (U_1 - i U_2) z \\ u &= U_1 \\ v &= U_2 \end{aligned}$$

This is, of course, equivalent to the oblique flow in II 6.

3. Rectilinear Flow Combined with Source. We may assume the source at the origin and the rectilinear flow along X and from right to left, *i. e.* along $-X$. Then from II 4 and II 8

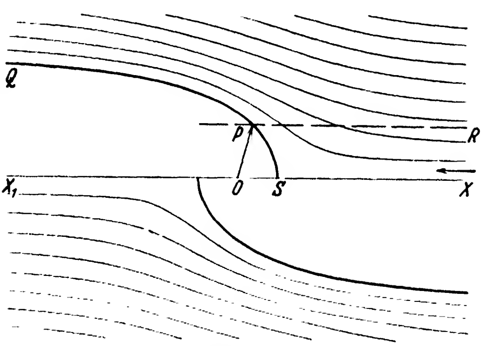

Fig. 14.

$$\varphi = -Ux + \mu \log r \qquad (3.1)$$
$$\psi = -Uy + \mu\theta \qquad (3.2)$$
$$w = -Uz + \mu \log z \qquad (3.3)$$

Combining likewise the individual values for u and v we have:

$$u = -U + \frac{\mu}{r} \cos\theta \qquad (3.4)$$
$$v = \frac{\mu}{r} \sin\theta \qquad (3.5)$$

For the stream-lines we have in (3.2) an equation in terms of the coordinates θ and y and with $\psi =$ *constant*, any line is readily traced. The character of this field is shown in Fig. 14.

In this case we have taken the axis of X as the common datum line and with the convention of sign as in Division A VIII 2 it is seen that, for any point P the flow along X between P and X will count as negative while that for the source will count as positive.

Consider the particular stream-line
$$\psi = -Uy + \mu\theta = 0$$

The significance of this zero value is then simply, that at some point P, for example, the flow outward from O, in the angle XOP is just equal to the flow along X between XO and RP. The actual direction and velocity of flow at P will, as we have seen, be determined by the direction and magnitude of the two component velocities at this point.

For the points fulfilling this condition we thus have the equation:
$$Uy = \mu\theta$$
or
$$y = \frac{\mu}{U}\theta \qquad (3.6)$$
from which the stream-line may be traced.

To find where this line cuts the axis of X, we have, for a point very near X, $y = r\,\theta$ and $r = x$.

Hence from (3.6) $x = \frac{\mu}{U}$

Otherwise we may locate this point by finding where on X the source velocity left to right just equals the velocity U along X, right to left. This gives the equation

$$\frac{m}{2\,\pi\,r} = U \quad \text{or} \quad r = x = \frac{m}{2\,\pi\,U} = \frac{\mu}{U} \quad \text{as before.}$$

The point S thus determined is known as a point of *stagnation*. It is a point where the velocity becomes zero due to the combination of two equal and opposite values of the component velocities.

The stream-line which contains a point of stagnation acts in general to separate the field of flow into two parts which thus flow as separate systems, having contact only along this line.

Thus in Fig. 14 if we take the line SPQ resulting from (3.6) and assume a symmetrical counterpart below X, we see that it will form a continuous line and act as a boundary or line of separation between the fluid issuing from O and that coming along X from the right. We have seen that stream-lines do not and cannot cross. No line in the infinite field on the right can cross SPQ to the left or inward, and no line on the left can cross SPQ to the right or outward. In consequence, considering only the flow above X, the fluid from the source O will remain below and on the left of the zero line SPQ, while that from the far distant sources of the rectilinear flow will remain above and on the right. Here again, assuming a thin sheet of two-dimensional flow, we may imagine a thin frictionless partition or barrier substituted for SPQ without in any way disturbing the flow from either set of sources. Thus considering only half the diagram, a thin solid body of the form SPQ might be substituted for the source, and the stream-line system lying above and on the right will then give the flow with an obstruction of this form placed in an infinite stream flowing along the upper side of the axis of X. Symmetry would give, of course, the same result below X as above and thus a symmetrical body formed of the combination of SPQ and its reflection in XX_1, placed in an indefinite stream flowing parallel to X would give a distribution of stream-lines shown by the system above X combined with its reflection in X.

Putting $\theta = \pi$ in (3.6), it is seen that the maximum breadth of the half obstruction X_1SPQ is $\mu\,\pi/U$ and which will be at $-\infty$ on X.

The direction of the curve SPQ at any point may be found by putting (3.6) in the form:

$$\frac{U\,y}{\mu} = \tan^{-1}\frac{y}{x}$$

whence
$$\frac{d\,y}{d\,x} = \frac{\mu\,y}{\mu\,x - (x^2 + y^2)\,U} = \frac{\mu\,y}{\mu\,x - U\,r^2}$$

At S where $\mu = Ux$ and $y = 0$, this becomes

$$\frac{dy}{dx} = -\frac{\mu y}{U y^2} = -\frac{\mu}{U y} = -\infty$$

Hence at S the curve is $\perp$ to X. Again for $x = -\infty$, $dy/dx = -0$. This shows the value of dy/dx always $(-)$ from $-\infty$ at S to -0 at the limit on the left. This all accords with the form of the curve as in Fig. 14.

4. Rectilinear Flow Combined with Sink. We take the rectilinear flow as in **3** and reverse the signs for the source, thus giving

$$\varphi = -Ux - \mu \log r$$
$$\psi = -Uy - \mu\theta$$
$$w = -Uz - \mu \log z$$
$$u = -U - \frac{\mu}{r} \cos\theta$$
$$v = -\frac{\mu}{r} \sin\theta$$

This field may be traced the same as for the source. It is symmetrical about X, the lower half being shown in Fig. 14 below XX. As might be anticipated, it is simply that for the source, as above XX, reversed in direction.

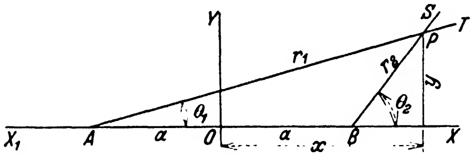

Fig. 15.

5. Two Sources of Equal Strength. Given two sources at A and B as in Fig. 15. This implies the adoption of the line AB or XX_1 as the common datum for the two sources. Then by the addition theorem and from II 8 we may write the functions as follows:

$$\varphi = \frac{\mu}{2} \log [(x+a)^2 + y^2] + \frac{\mu}{2} \log [(x-a)^2 + y^2] \tag{5.1}$$

or

$$\varphi = \mu \log r_1 r_2 \tag{5.2}$$

$$\psi = \mu \left[\tan^{-1} \frac{y}{x+a} + \tan^{-1} \frac{y}{x-a} \right] \tag{5.3}$$

and

$$\psi = \mu (\theta_1 + \theta_2) \tag{5.4}$$

$$w = \mu \log (z+a)(z-a) = \mu \log (z^2 - a^2) \tag{5.5}$$

From (5.1) we have

$$u = \frac{\partial \varphi}{\partial x} = \frac{\mu (x+a)}{(x+a)^2 + y^2} + \frac{\mu (x-a)}{(x-a)^2 + y^2} = \mu \left[\frac{x+a}{r_1^2} + \frac{x-a}{r_2^2} \right] \tag{5.6}$$

$$v = \frac{\partial \varphi}{\partial y} = \frac{\mu y}{(x+a)^2 + y^2} + \frac{\mu y}{(x-a)^2 + y^2} = \mu y \left[\frac{1}{r_1^2} + \frac{1}{r_2^2} \right] \tag{5.7}$$

It may be of interest to check the form of (5.5) by differentiation of w for u and v. Thus

$$\frac{dw}{dz} = \frac{\mu}{z+a} + \frac{\mu}{z-a} = \frac{\mu}{(x+a) + iy} + \frac{\mu}{(x-a) + iy}$$

or
$$\frac{dw}{dz} = \frac{\mu\,[(x+a) - i\,y]}{(x+a)^2 + y^2} + \frac{\mu\,[(x-a) - i\,y]}{(x-a)^2 + y^2}$$

or
$$\frac{dw}{dz} = u - i\,v = \frac{\mu\,(x+a)}{r_1^2} + \frac{\mu\,(x-a)}{r_2^2} - \frac{\mu\,i\,y}{r_1^2} - \frac{\mu\,i\,y}{r_2^2}$$

whence
$$u = \mu\left[\frac{(x+a)}{r_1^2} + \frac{(x-a)}{r_2^2}\right]$$
$$v = \mu\,y\left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right]$$

all as in (5.6) (5.7).

Instead of this immediate derivation of the functions φ, and ψ by combination from II 8, it will be instructive to repeat a part of the process with a little more of individual detail.

The two individual stream functions, each referred to its own center will be:
$$\psi_1 = \mu\,\theta_1 = \text{constant}$$
$$\psi_2 = \mu\,\theta_2 = \text{constant}$$

Hence for the combined field we shall have
$$\psi = \mu\,[\psi_1 + \psi_2] = \mu\,(\theta_1 + \theta_2) = \text{constant} \tag{5.8}$$

This, it is seen, is the equivalent of (5.3), the latter being expressed in rectangular coordinates relative to O as center. Equation (5.8) alone would enable us readily to trace the stream-lines. Thus $X\,B$, Fig. 16, will correspond to

$\theta_1 + \theta_2 = 0$, $B\,A$ to $\theta_1 + \theta_2 = \pi$ and $A\,X_1$ to $\theta_1 + \theta_2 = 2\,\pi$

Again $O\,Y$ will correspond to $\theta_1 + \theta_2 = \pi$ and $O\,Y_1$ to $\theta_1 + \theta_2 = 3\,\pi$. The two axes X and Y will therefore belong to the stream-line system and any other line may be traced in geometrically by laying off from B a line $B\,P$ at an angle θ_2 and then from A, a line $A\,P$ at an angle $(\psi - \theta_2) = \theta_1$. Where these lines intersect will be a point on the stream-line. The general character of the resulting stream-line field is shown in Fig. 16.

Turning now to the stream-line field as expressed analytically in (5.3), we have:
$$\tan\frac{\psi}{\mu} = \frac{\dfrac{y}{x+a} + \dfrac{y}{x-a}}{1 - \dfrac{y^2}{x^2 - a^2}}$$

whence
$$(x^2 - y^2 - a^2)\tan\frac{\psi}{\mu} - 2\,x\,y = 0$$

As an exercise in analytical geometry, this equation is readily seen to give, for ψ *constant*, a hyperbola, asymptotic to two inclined lines through O. For a series of values of ψ from 0 to the strength of the source, we shall have a series of hyperbolae as indicated in Fig. 16.

Attention may be called to certain features of this field of flow.

(1) The stream-lines all pass through either one source or the other.

(2) The point of stagnation must lie at the point O where the two velocities from A and B are equal and opposite. The stream-line which passes through this point is, as we have seen, the axis YOY_1. It is also seen that for this line we have

$$\psi = \psi_1 + \psi_2 = \mu(\theta_1 + \theta_2) = \pi\mu = \frac{m}{2}$$

As in 3 this line acts as the ultimate or limiting stream-line, common to and separating the two parts of the joint field. Hence as in 3 we imagine this ultimate stream-line replaced by a rigid smooth barrier without in the least interfering with the flow from either source. We may then suppress the flow from either source without in any way disturbing that from the other. We thus arrive at the field on one side of Y as representing the flow for a source placed near an indefinite straight barrier, all, of course, for two-dimensional flow.

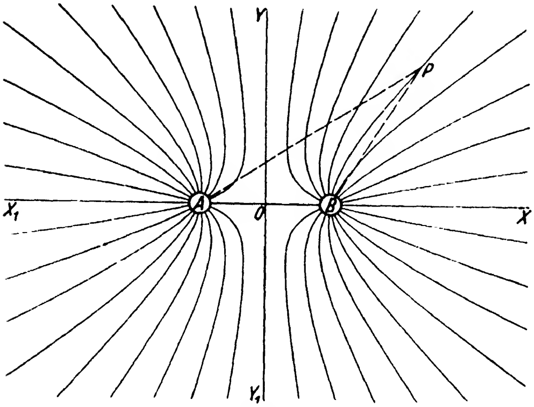

Fig. 16.

6. Two Sinks of Equal Strength. By a complete reversal of sign, a source becomes a sink, and two sources become two sinks. The lines of flow are reversed in direction, but otherwise unchanged.

Likewise, and for the same reason as with the source, the common stream-line YY_1 may be considered as a solid barrier, and the field on either side then becomes the field of flow for a sink in the vicinity of a solid barrier.

7. Two Sources of Unequal Strength. If we designate the two strengths by m_1 and m_2 then II 8 gives,

$$\varphi = \mu_1 \log r_1 + \mu_2 \log r_2 \tag{7.1}$$

$$\psi = \mu_1\theta_1 + \mu_2\theta_2 \tag{7.2}$$

$$w = \mu_1 \log(z + a) + \mu_2 \log(z - a) \tag{7.3}$$

The velocities, u and v follow by differentiation, or may be written directly from 5 by putting μ_1 and μ_2 instead of μ throughout.

Assuming m_1 at A and m_2 at B with $m_1 < m_2$, the stream-line field is shown in Fig. 17.

The line XB corresponds to $\psi = 0$, the line BA to $\psi = \pi\mu_2$ and the line AX_1 to $\psi = \pi(\mu_1 + \mu_2)$.

For the point of stagnation we must have the point Q where the two velocities from A and B are equal and opposite. Hence at Q

$$\frac{\mu_2}{r_2} = \frac{\mu_1}{r_1} \quad \text{or} \quad \frac{r_1}{r_2} = \frac{\mu_1}{\mu_2}$$

That is, the point Q divides AB in the ratio of the factors μ_1 and μ_2 or of the strengths m_1 and m_2. Likewise for this point and hence for the stream-line passing through it,

$$\psi = \pi \mu_2 = \frac{m_2}{2}$$

If we should take XX_1 as the positive direction on X, counting θ_1 and θ_2 from AX_1 and BX_1, the line PQ would be the same, but its equation would now be $\psi = \pi \mu_1 = \frac{m_1}{2}$

For two sinks of unequal strength, the equations will result immediately by making μ_1 and μ_2 negative in (7.1), (7.2), (7.3).

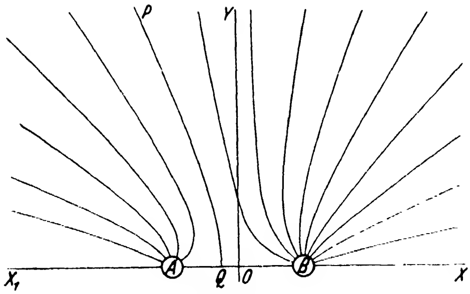

Fig. 17.

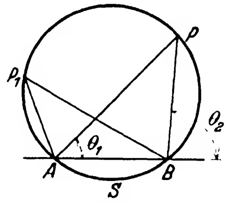

Fig. 18.

8. Source and Sink of Equal Strength. We derive the various functions and expressions for this case from those for two sources as in 5, by taking the μ for A, for example, negative. We thus have:

$$\varphi = \frac{\mu}{2} \log \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \tag{8.1}$$

or

$$\varphi = \mu \log \frac{r_2}{r_1} \tag{8.2}$$

$$\psi = \mu \left[\tan^{-1} \frac{y}{x-a} - \tan^{-1} \frac{y}{x+a} \right] \tag{8.3}$$

or

$$\psi = \mu (\theta_2 - \theta_1) \tag{8.4}$$

$$w = \mu \log \frac{z-a}{z+a} \tag{8.5}$$

$$u = \mu \left[\frac{x-a}{r_2^2} - \frac{x+a}{r_1^2} \right] \tag{8.6}$$

$$v = \mu y \left[\frac{1}{r_2^2} - \frac{1}{r_1^2} \right] \tag{8.7}$$

The geometric character of the stream-lines is readily seen as follows:

In Fig. 18 $(\theta_2 - \theta_1)$ = angle at P. But for a stream-line this will be constant [see (8.4)]. Hence P will lie on the circumference of a circle passing through PBA, for, in such case, the chord AB will cut off a constant arc ASB and hence give a constant value to the angle at P. The stream-lines are therefore circles all passing through A and B, and as indicated in Fig. 19.

Analytically this develops as follows:

From (8.3) we have $tan \frac{\psi}{\mu} = \dfrac{\dfrac{y}{x-a} - \dfrac{y}{x+a}}{1 + \dfrac{y^2}{x^2 - a^2}}$

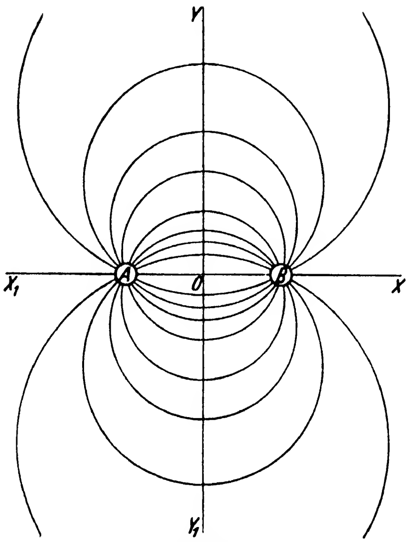

Fig. 19.

This reduces directly to

$$(x^2 + y^2 - a^2)\, tan \frac{\psi}{\mu} - 2ay = 0$$

Or putting

$$K = tan \frac{\psi}{\mu} = constant,$$

$$x^2 + y^2 - \frac{2ay}{K} = a^2 \qquad (8.8)$$

This is readily seen to be a circle with center on Y at a distance of a/K above O and with radius $a \div sin\,(\psi/\mu)$.

In this field, the following points may be noted:

(1) The line XB represents the stream-line $\psi = 0$.

(2) The stream-lines are all circles and hence closed paths, the limit being the axis of X which returns on itself through infinity.

(3) There is no point of stagnation and no line of separation as in 3, 5 and 7.

9. Source and Sink of Unequal Strength. The equations for this case result directly from those for 7 by taking μ_2 for example, (—) instead of (+). We thus have

$$\varphi = \mu_1\, log\, r_1 - \mu_2\, log\, r_2 \qquad (9.1)$$

$$\psi = \mu_1 \theta_1 - \mu_2 \theta_2 \qquad (9.2)$$

$$w = \mu_2\, log\,(z + a) - \mu_2\, log\,(z - a) \qquad (9.3)$$

The component velocities u and v may be derived by differentiation of φ or ψ, or written directly from 8 by putting μ_1 and μ_2 instead of μ throughout.

Assuming m_1 and m_2 as in Fig. 20 with $m_1 > m_2$, the diagram shows the general character of the upper half of the stream-line field with,

of course, the other half symmetrical below the axis XX_1. Regarding this field, the following points may be noted:

(1) The stream-lines are partly closed curves and partly open.

(2) The closed paths represent the flow $m_2 =$ a part of the flow issuing from B and the total flow absorbed by A.

(3) There is a point of stagnation at D where $-x = (m_1 + m_2)a/(m_1 - m_2)$ and a line BCD separating these two flows.

10. Doublet. The product of the strength factor $\mu = m/2\pi$ and the distance apart of a source and a sink $(2a)$ has a certain analogy to a moment. Using this form for convenience we have

Moment $= 2a\mu$

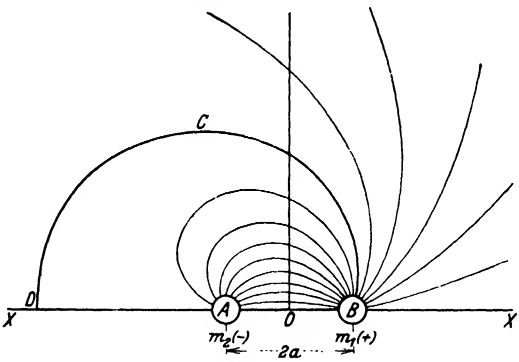

Fig. 20.

Now imagine the distance $2a$ to be continuously decreased while μ increases correspondingly so that the product or moment as we have termed it, remains constant. Such a combination at the limit is called a *doublet*.

The various functions for a doublet are derived from those for a source and sink as follows:

In 8 for a source sink combination, we shall have, at the limit, μ very large and a very small. The angle $\psi/\mu = (\theta_2 - \theta_1)$ in (8.4) becomes very small and we may write (8.4) $\psi/\mu = tan\,(\theta_2 - \theta_1)$. If we then expand this by the formula for the tangent of the difference of two angles and insert the values of the tangents from (8.3), we shall have:

$$\psi = \frac{2a\mu y}{x^2 + y^2 - a^2}$$

But $2a\mu$ is the constant moment of the combination or of the doublet, and which we may denote by M. Likewise at the limit a will disappear from the denominator and we thus have

$$\psi = \frac{My}{x^2 + y^2} = \frac{My}{r^2} \tag{10.1}$$

or
$$x^2 + y^2 - \frac{M}{\psi}\, y = 0$$

or
$$x^2 + \left(y - \frac{M}{2\psi}\right)^2 = \left(\frac{M}{2\psi}\right)^2 \tag{10.2}$$

Equation (10.2) is seen to be the equation to a series of circles all passing through the origin as in Fig. 21. This result might have been

anticipated from Fig. 19, for if the points A and B are drawn indefinitely near together, Fig. 19 will become transformed into Fig. 21.

We may next derive φ as follows:

$$\frac{\partial \varphi}{\partial y} = \frac{-\partial \psi}{\partial x} = \frac{2 M x y}{(x^2 + y^2)^2}$$

Integrating this with reference to y, we have

$$\varphi = \frac{-M x}{x^2 + y^2} = -\frac{M x}{r^2} \tag{10.3}$$

Again for the components u and v we have

$$u = \frac{\partial \varphi}{\partial x} = \frac{M (x^2 - y^2)}{r^4} = \frac{M}{r^2} \cos 2\theta \tag{10.4}$$

$$v = \frac{\partial \varphi}{\partial y} = \frac{2 M x y}{r^4} = \frac{M}{r^2} \sin 2\theta \tag{10.5}$$

$$V = \sqrt{u^2 + v^2} = \frac{M}{r^2} \tag{10.6}$$

For the function w we have

$$w = \varphi + i\psi = \frac{-M x}{r^2} + \frac{i M y}{r^2}$$

or

$$w = -M \frac{x - i y}{x^2 + y^2} = \frac{-M}{x + i y} = -\frac{M}{z} \tag{10.7}$$

This may also be derived directly from (8.5) as follows: Thus,

$$w = \mu \left[\log (z - a) - \log (z + a)\right]$$

But by Taylor's Theorem with a very small and dropping terms in a^2 and higher, we have

$$w = \mu \left[\left(\log z - \frac{a}{z}\right) - \left(\log z + \frac{a}{z}\right)\right]$$

or

$$w = -\frac{2 \mu a}{z} = -\frac{M}{z}$$

11. Combination of Sources and Sinks Distributed Along a Line. Analytically this problem is simply an extention of 5—8. We have simply to express the individual values of the functions φ and ψ, sum, having due regard to the signs (+ for source, — for sink) and express the final in the form

$$\Phi = \Sigma \varphi$$
$$\Psi = \Sigma \psi$$

Actually the analytical work soon becomes complicated and tedious. Numerically, however, the values of φ and ψ are readily found for any specified point, or for a series of points, by a simple addition of the values for the individual elements. By suitable graphical methods, such points may be located for a series of constant values of Φ or Ψ and thus equipotentials or stream-lines determined as may be desired. Thus in Fig. 22 suppose we have eight sources and sinks distributed as indicated by the dots, the sources marked + and the sinks —. Let A, B, C, etc.

be a series of points spaced on a line LM. Then the numerical values of ψ for these points are readily determined. Let these values be laid off on a line representing LM, see Fig. 23, thus giving the curve JK. It is then an easy matter to take points A_1, B_1, etc. on this curve, such that the values of Ψ progress in uniform manner. These points projected back to LM will then give the locations for a series of values of Ψ with regular increasing values.

In the same way, similar points for the same series of values of Ψ may be located on lines drawn in any manner in the field as desired, parallel to Y or otherwise. Thus from the right hand source, we might draw radiating lines as indicated in the diagram. The whole procedure may be summarized as follows:

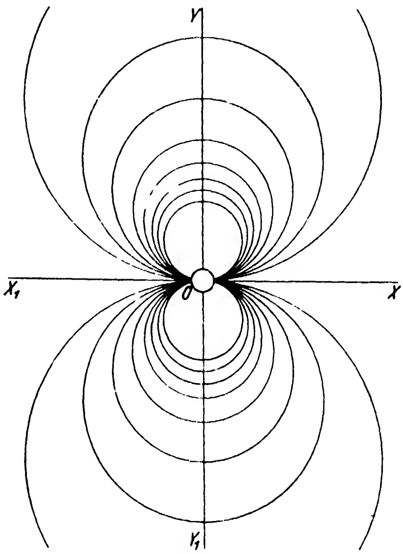

Fig. 21.

(1) Take any line anywhere in the field.

(2) Locate thereon a series of points, spaced in any manner desired.

(3) Find values of Ψ (or of Φ) for these points.

(4) Plot such values, as an auxiliary diagram, along an axis representing this line.

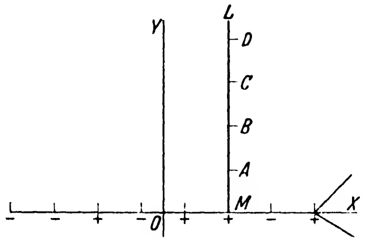

Fig. 22.

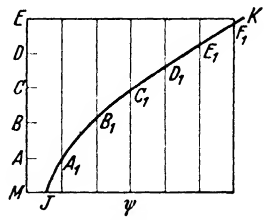

Fig. 23.

(5) Locate the points on this auxiliary curve corresponding to equidistant or to specially selected values of Ψ (or Φ) and project such points back to the axis.

(6) Transfer the points thus found on the axis to the line in the field, marking them for identification.

(7) Carry out this process to an extent sufficient to give a field of points adequate for the purpose desired, and then draw a line through a series of points having the same number or value of Ψ (or Φ).

(8) The result will be a stream-line (or equipotential).

Others are drawn in similarly and to the extent desired.

For the case where the elements are doublets instead of individual sources or sinks, the general method is entirely similar and need not be considered in detail.

Many short cuts and useful variations in detail will suggest themselves in connection with specific problems, but these may properly be left to the interested reader.

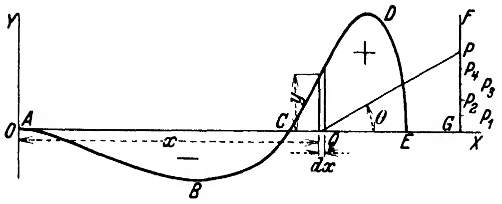

Fig. 24.

12. Field of Flow for a Continuous Source and Sink Distribution Along a Line. The limit of distribution, in the problems immediately preceding is found in the assumption of a continuous source and sink distribution along a line. Thus in Fig. 24 suppose a continuous source distributed along the line CE, the law of the distribution of strength being given by the ordinates of the curve CDE, and similarly with ABC for a continuous sink. Such a distribution may have any degree of complexity of distribution and form, see Fig. 25.

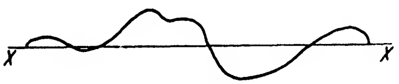

Fig. 25.

A physical picture of such a flow may be obtained by supposing AE to represent a narrow slot with width varying according to the ordinates of the curve $ABCDE$, the flow to be outward along CE and inward along AC.

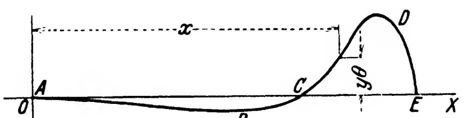

Fig. 26.

Let dx denote an element of length along the line CE, y the corresponding ordinate and m a constant. Then $m\,y\,dx$ will denote the strength of the elementary source represented by dx. It follows that the integral of $m\,y\,dx$ will give the total source strength. This is obviously m multiplied into the area CDE. In the same manner m times the area ABC will be the total sink strength. If these two areas are equal, the source and sink strengths are equal and the system is balanced as a whole. If they are unequal, the source and sink strengths are unbalanced with a general result similar to that of 9.

The stream-lines for such a continuous source and sink distribution may be determined by an extension of methods indicated in 11. Thus for the point P Fig. 24, in order to find the value of the stream function ψ we may proceed as follows:

For an element of the source line at Q the strength is $m\,y\,dx$ and the angle for the point P is $XQP = \theta$. Hence we shall have (see II 8)

$$\psi = \frac{m}{2\pi}\int y\,\theta\,dx$$

where the integration is carried from A to E and with due regard to the negative sign of y for the sink part of the line. In effect this is simply the area of a new curve laid off from AE as the axis, and of which the ordinates are proportional to $y\theta$. We may, therefore, simply lay off, as an auxiliary diagram, a curve with ordinates $= y\theta$ on the line representing AE as base. See Fig. 26.

The area of this curve is then to be found by planimeter or by numerical integration, multiplied by $m/2\pi$ and the result will be the value of ψ for the point P.

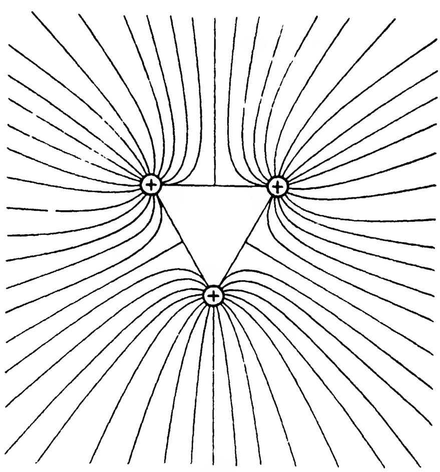

Fig. 27.

In this general manner values of ψ may be found for series of points distributed along lines or over the field in any manner desired. Such values then plotted in auxiliary diagrams, in manner similar to that of **11**, will permit the location of points in the field for uniform series of values of ψ and through which, as in **11**, stream-lines may be drawn.

For equipotentials, the general method is entirely similar. We have now, however, to deal with the general function $\varphi = (m/2\pi)\log r$ (see II 8) and the value of φ will be given by

$$\varphi = \frac{m}{2\pi}\int y\log r\,dx$$

We have, therefore, in such case to lay off a curve on the base line AE, of which the ordinates represent $y \log r$ (where r denotes a distance such as QP), find its area, and proceed generally as for the function ψ.

13. Combination of Sources and Sinks Distributed in any Manner in A Plane. For a distribution of sources and sinks over a plane, there is, in general, no single line common as a stream-line to all of the component elements, and hence we cannot, at a single operation (as in **11** or **12**) find a single resultant value of ψ for a given point in the field.

The problem may, however, be dealt with graphically by a continued application of the principles of **1**. The various elements may be combined in pairs, and these combination in pairs until a final result is reached.

In general the work required to carry out such a program of combinations would be prohibitive, but it is not without interest to note

the possibility of dealing with such problems in case their importance should justify the labor involved.

In fact, to any degree of accuracy desired, continuous distributions along curved lines may be treated in the same general manner. The curve must be divided up into elements which may be considered as practically straight lines, and these dealt with either as short straight lines after the manner of 12 or less accurately as point elements with strengths representing the elementary line strengths. Here again, *a fortiori*, the labor required would be excessive, and beyond justification in any ordinary case.

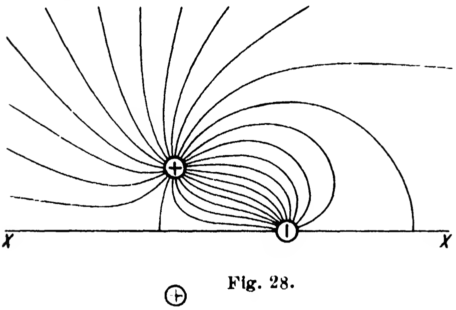

Fig. 28.

As a very simple illustration, Fig. 27 shows the field for the combination of three equal sources at the angles of an equilateral triangle, and Fig. 28 the result when one of the elements is a sink and the other two are sources.

CHAPTER V

COMBINATION FIELDS OF FLOW CONTINUED—KUTTA-JOUKOWSKI THEOREM

1. Rectilinear Flow with Source and Sink of Equal Strength. Taking, as before, the rectilinear flow parallel to $-X$ we have from II 4; IV 8:

$$\varphi = -Ux + \mu \log \frac{r_2}{r_1} \tag{1.1}$$

$$\psi = -Uy + \mu(\theta_2 - \theta_1) \tag{1.2}$$

$$w = -Uz + \mu \log \frac{z-a}{z+a} \tag{1.3}$$

$$u = -U + \mu \left[\frac{x-a}{r_2^2} - \frac{x+a}{r_1^2}\right] \tag{1.4}$$

$$v = \mu y \left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] \tag{1.5}$$

The general character of this field is shown in Fig. 29.

For the point of stagnation we must have the point P on X, where the velocity to the right due to the combination of the source and sink will just equal the velocity U to the left, with of course, a symmetrical point at Q. This will develop from (1.4) by putting $u = 0$, $r_2 = (x - a)$ and $r_1 = (x + a)$. Making these substitutions and reducing, we find

$$x = \pm \sqrt{\frac{2a\mu}{U} + a^2}$$

At the points P and Q, we have $\psi = 0$ [see (1.2)]. Hence the stream-line which passes through P and Q will be given by putting $\psi = 0$, or *vice versa*, the line $\psi = 0$, in this case, contains also the points of stagnation. For this line, we have from (1.2) in terms of x and y:

$$U\,y = \mu\,(\theta_2 - \theta_1) = \mu\left(\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right) \tag{1.6}$$

and from which the curve may be traced. The equation is, however, readily put in a more convenient form:

$$\tan\frac{U\,y}{\mu} = \frac{2\,a\,y}{x^2+y^2-a^2} = \frac{2\,a\,y}{r^2-a^2} \tag{1.7}$$

The general character of the curve is an oval as in Fig. 29 with points on X at P and Q as given above. For the points on Y, we have, from the geometry of the figure:

$$(\theta_2 - \theta_1) = 2\tan^{-1}\frac{a}{y}$$

From (1.6) this gives

$$\tan\frac{U\,y}{2\,\mu} = \frac{a}{y} \tag{1.8}$$

from which y may be found by trial and error.

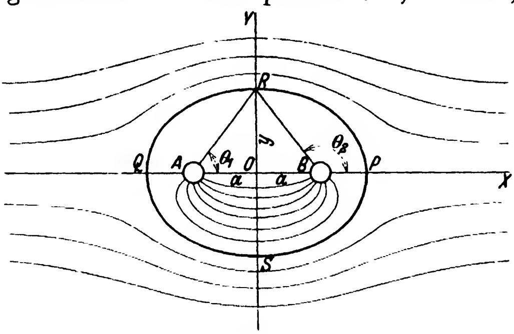

Fig. 29.

As in other cases, the stream-line through the points of stagnation, P and Q (in this case the line $\psi = 0$), will separate the fluid of the source-sink system from that of the rectilinear flow system. Likewise, the stream-line system, outside this boundary curve $P\,R\,Q\,S$, will give the distribution of flow in a thin flat sheet flowing from right to left and around an obstacle of the form given by the curve. And likewise, the lines within $P\,R\,Q\,S$ give the distribution of the stream-line system for a source-sink combination placed within a boundary of the form given by the curve.

2. Rectilinear Flow with Doublet—Infinite Flow About a Circle. With the same disposition of rectilinear flow as in **1** we have the combination of the functions of II **4** and IV **10** as follows:

$$\varphi = -\,U x - \frac{M\,x}{r^2} \tag{2.1}$$

$$\psi = -\,U y + \frac{M\,y}{r^2} \tag{2.2}$$

$$w = -\,U z - \frac{M}{z} \tag{2.3}$$

$$u = -\,U + \frac{M\,(x^2 - y^2)}{r^4} \tag{2.4}$$

$$v = \frac{2\,M\,x\,y}{r^4} \tag{2.5}$$

From the line $\psi = 0$ we have $\quad U = \dfrac{M}{r^2}$

or
$$x^2 + y^2 = \frac{M}{U}$$

This is a circle with O as center and radius $\sqrt{M/U}$. Denoting this radius by a we have:
$$a^2 = \frac{M}{U} \quad \text{or} \quad a = \sqrt{\frac{M}{U}} \tag{2.6}$$

The general character of the fields of flow may be found from (2.2) and are shown in Fig. 30. Here again, the circle $A\,B\,C\,D$ will act as a boundary between the fluid from the doublet and that forming the rectilinear flow, and the distribution of stream-lines outside the circle will, therefore, represent the disposition for the case of an indefinite rectilinear flow (assumed from right to left) around a circular obstruction as represented by $A\,B\,C\,D$. On the other hand, the field within the circle represents the lines of flow for a doublet surrounded by a circular barrier represented by $A\,B\,C\,D$.

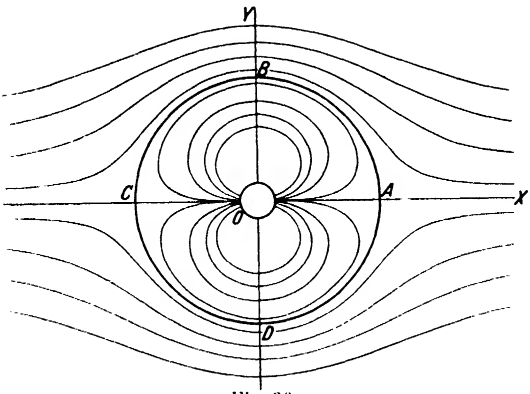

Fig. 30.

Cases of this character are often more conveniently dealt with by means of polar or mixed coordinates. We substitute $M = U a^2$ and thus find:
$$\varphi = -\,U x\left(1 + \frac{a^2}{r^2}\right) = -\,U\left(r + \frac{a^2}{r}\right)\cos\theta \tag{2.7}$$
$$\psi = -\,U y\left(1 - \frac{a^2}{r^2}\right) = -\,U\left(r - \frac{a^2}{r}\right)\sin\theta \tag{2.8}$$
$$w = -\,U\left(z + \frac{a^2}{z}\right) \tag{2.9}$$

Then as before, using n and c for the components along and $\perp$ to the radius we shall have:
$$n = \frac{\partial\varphi}{\partial r} = -\,U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{2.10}$$
$$c = \frac{\partial\varphi}{r\,\partial\theta} = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \tag{2.11}$$

Also transforming (2.4) and (2.5) into polar coordinates we have:

$$u = -U + \frac{U a^2}{r^2} (\cos^2 \theta - \sin^2 \theta) = -U + \frac{U a^2}{r^2} \cos 2\theta \qquad (2.12)$$

$$v = -\frac{2 U a^2}{r^2} \sin \theta \cos \theta = \frac{U a^2}{r^2} \sin 2\theta \qquad (2.13)$$

It will be noted that $n = 0$ when $r = a$, that is, for all points on the circumference of the circle $ABCD$. Likewise for such points,

$$c = 2U \sin\theta = \text{total velocity} \qquad (2.14)$$

This is a maximum for $\theta = \pm 90^0$ and becomes zero for $\theta = 0$ and $\theta = \pi$ at points A and C.

The points of stagnation are, in this case of course, located at A and C, given either as points on the line $\psi = 0$, or from (2.12) by putting $u = 0$, $\theta = 0$ and $r = x$, whence $x = \pm a$.

The total velocity at any point will be found from

$$V^2 = n^2 + c^2$$

Carrying out these operations and reducing, we find:

$$V^2 = U^2 \left[1 + \frac{a^4}{r^4} - 2 \frac{a^2}{r^2} \cos 2\theta\right] \qquad (2.15)$$

For points on $ABCD$, this becomes

$$V^2 = c^2 = 4 U^2 \sin^2\theta \text{ [as in (2.14)]} \qquad (2.16)$$

The pressure at any point is found from the Bernoulli equation I (6.6) and gives here

$$p_0 + \frac{1}{2} \varrho U^2 = p + \frac{1}{2} \varrho V^2$$

where the left hand side is the total head at a far distant point, made up of the field pressure p_0 and the velocity head $(1/2) \varrho U^2$. We shall have then in general,

$$p = p_0 - \frac{1}{2} \varrho U^2 \left[\frac{a^4}{r^4} - 2 \frac{a^2}{r^2} \cos 2\theta\right] \qquad (2.17)$$

For points on the circle $ABCD$ this becomes

$$p = p_0 - \frac{1}{2} \varrho U^2 (1 - 2 \cos 2\theta)$$

or

$$p = p_0 + \frac{1}{2} \varrho U^2 (1 - 4 \sin^2 \theta) \qquad (2.18)$$

Either of these forms shows p to be symmetrical with reference to X and Y and hence that the resultant p over the circumference of such a circle will be zero. It follows that the resultant force on a thin circular obstacle in a sheet of flow as here assumed will be zero. This, of course, will hold for a fluid having the properties assumed—that is for the perfect non-viscous fluid. For actual fluids with viscosity, the case will be quite different as will appear at a later point.

If we suppose the field of flow in Fig. 30 revolved through + 90 deg., the results will give the flow in the direction of $-Y$ past a circle at

the origin. The functions φ, ψ and w for this flow follow directly from (2.7) and (2.8) by putting $+y$ for $+x$ and $-x$ for $+y$. We thus find, calling the velocity V,

$$\varphi = -Vy\left(1+\frac{a^2}{r^2}\right) = -V\left(r+\frac{a^2}{r}\right)\sin\theta \qquad (2.19)$$

$$\psi = Vx\left(1-\frac{a^2}{r^2}\right) = V\left(r-\frac{a^2}{r}\right)\cos\theta \qquad (2.20)$$

$$w = iV\left(z-\frac{a^2}{z}\right) \qquad (2.21)$$

If now we assume a field velocity V inclined at an angle α to the axis of x (downward and inward) then $-V\cos\alpha$ will take the place of $-U$ in Eqs. (2.7), (2.8), and $-V\sin\alpha$ that of V in Eqs. (2.19), (2.20). Making these changes and adding the two component parts of the functions φ and ψ we shall have, for the inclined flow

$$\varphi = -V\left(1+\frac{a^2}{r^2}\right)(x\cos\alpha + y\sin\alpha) \qquad (2.22)$$

$$\psi = -V\left(1-\frac{a^2}{r^2}\right)(x\sin\alpha + y\cos\alpha) \qquad (2.23)$$

For the potential function we then find,

$$w = \varphi + i\psi = -V\left[(\cos\alpha - i\sin\alpha)z + (\cos\alpha + i\sin\alpha)\frac{a^2}{z}\right]$$

or [see Divison A I 5 (b) (c)]

$$w = -V\left(e^{-i\alpha}z + \frac{a^2 e^{i\alpha}}{z}\right) \qquad (2.24)$$

3. Rectilinear Flow with Any of the Source and Sink Distributions of IV, 11, 12, 13. In 1 above we have seen that if the source and sink strengths are equal, the combination with an infinite straight line flow will give, for the line through the point of stagnation, a closed stream-line separating one part of the field from the other; and it is evident that if they are unequal, the corresponding result will be an open stream-line. The same is true for the more complex combinations of IV **11**, **12** and **13**. If the aggregate source strength is equal to the aggregate sink strength, the line will be closed; otherwise, open.

To find the function ψ for the combination of straight line flow with any of the complex source and sink distributions of the sections noted, we have first to find in the same general manner as in IV **11**, **12**, the total ψ function for the source sink distribution. For the infinite stream, the value is $\psi = -Uy$. Hence if we add this to the value found otherwise for each point in the field, we shall have the resultant value of ψ for the combination. These values then treated graphically by the aid of auxiliary diagrams, will serve to give points on a regular system of stream-lines for the final resultant field.

Otherwise we may plot graphically the field for the source and sink combination as in IV 11 or IV 12, then draw in the lines for the straight line flow and continue by the general method of IV 1.

The same general procedure holds, of course, for equipotentials as for stream-lines.

4. Indefinite Stream with Circular Obstacle Combined with Vortex Flow—Indefinite Flow with Circulation. This combination results from three elementary forms of flow:

(1) Rectilinear indefinite flow.

(2) Doublet flow.

(3) Vortex flow, the center of the vortex being taken at the doublet.

The combination of (1) and (2) is given in 2 and to find the combination of all three, we have only to combine the results of that section with III 3.

Thus,
$$\varphi = -Ux - \frac{Mx}{r^2} + k \tan^{-1}\frac{y}{x} \tag{4.1}$$
$$\psi = -Uy + \frac{My}{r^2} - k \log\frac{r}{a} \tag{4.2}$$
$$w = -Uz - \frac{M}{z} - ik \log z \tag{4.3}$$
$$u = -U + \frac{M(x^2 - y^2)}{r^4} - \frac{ky}{r^2} \tag{4.4}$$
$$v = \frac{2Mxy}{r^4} + \frac{kx}{r^2} \tag{4.5}$$

In these equations, the a of (4.2) is, of course, the radius of the circular obstacle or boundary, and equal to $\sqrt{M/U}$. See (2.6).

Transforming to polar or to mixed coordinates as in 2 we have:
$$\varphi = -Ux\left(1 + \frac{a^2}{r^2}\right) + k\theta = -U\left(r + \frac{a^2}{r}\right)\cos\theta + k\theta \tag{4.6}$$
$$\psi = -Uy\left(1 - \frac{a^2}{r^2}\right) - k\log\frac{r}{a} = -U\left(r - \frac{a^2}{r}\right)\sin\theta - k\log\frac{r}{a} \tag{4.7}$$
$$w = -U\left(z + \frac{a^2}{z}\right) - ik\log z \tag{4.8}$$
$$u = -U + \frac{Ua^2}{r^2}\cos 2\theta - \frac{k}{r}\sin\theta \tag{4.9}$$
$$v = U\frac{a^2}{r^2}\sin 2\theta + \frac{k}{r}\cos\theta \tag{4.10}$$

The velocities along and perpendicular to the radius follow from (4.6)
$$n = -U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{4.11}$$
$$c = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{k}{r} \tag{4.12}$$

The total velocity V will be given then by
$$V^2 = n^2 + c^2 \quad \text{or} \quad u^2 + v^2$$

Choosing the first of these two sets we have

$$\left.\begin{aligned} V^2 = U^2\left(1 - \frac{a^2}{r^2}\right)^2 \cos^2\theta + U^2\left(1 + \frac{a^2}{r^2}\right)^2 \sin^2\theta + \\ + \frac{2kU}{r}\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{k^2}{r^2} \end{aligned}\right\} \tag{4.13}$$

This expression admits of some reduction in form, but having in view the uses required at a later point, there is nothing to be gained from such change, and the expression may be conveniently employed in the form as above.

For the general stream-line we have from (4.7)

$$y = -\frac{\psi + k \log \frac{r}{a}}{U\left(1 - \frac{a^2}{r^2}\right)} \tag{4.14}$$

From the character of this equation, it is clear that for a given value of r, the resultant value of y will apply on either side of the axis YY. Hence the field of flow will be symmetrical about YY.

The character of the resulting field of flow is shown in Fig. 31. From its importance in later studies, this field distribution will justify a more detailed examination than some of the cases noted previously.

For reasons of symmetry, however, we may confine our attention to the part lying on either side of the axis of symmetry YY.

To find where the stream-lines are horizontal we have only to put $v = 0$. This gives:

$$\frac{2Mxy}{r^4} + \frac{kx}{r^2} = 0 \quad \text{or} \quad \frac{2Ua^2xy}{r^4} + \frac{kx}{r^2} = 0$$

whence dividing by $1/r^2$ we have

$$\frac{1}{r^2} = 0 \quad \text{or} \quad r = \infty$$

Then dividing by x we have $x = 0$ and finally we have left:

$$2Ua^2y = -r^2k = -k(x^2 + y^2)$$

or

$$x = \pm\sqrt{-\frac{2Ua^2y}{k} - y^2} \tag{4.15}$$

The value of $r = \infty$ means all points at ∞, where the lines become parallel to X. It also appears from (4.9) that at $r = \infty$, $u = -U$ and thus the only velocity sensible at far distant points is the rectilinear component U, as we should expect from the nature of the other components.

The value of $x = 0$ gives the axis of Y, on which the lines are all parallel to X.

The value of x in (4.15) gives points such as R and S. These are possible when, numerically,

$$y < \frac{2Ua^2}{k} \quad \text{or} \quad < \frac{2Ua}{c_0}$$

where c_0 = circumferential velocity of vortex at radius a. When $y = 2Ua/c_0$ the three points R, S, T, come together on Y.

To find the point of stagnation P, we note that, on the circumference of the circle of radius a, the velocity n is always zero [see (4.11)]. To find where the total velocity is zero, we have, therefore, simply to find where the velocity c vanishes. To this end we put $r = a$ and $c = 0$ in (4.12) and find

$$\sin\theta = -\frac{k}{2aU} \quad \text{or} \quad y = -\frac{k}{2U} \tag{4.16}$$

Turning now to (4.7) it is seen that $r = a$ gives a zero value for ψ regardless of the value of θ. That is, ψ is zero for every point on the circle. Hence for any point of stagnation on the circle, the stream-line passing through such point must have as its equation $\psi = 0$. But this is not all of the line $\psi = 0$. From (4.7) we have, for the general equation to such line:

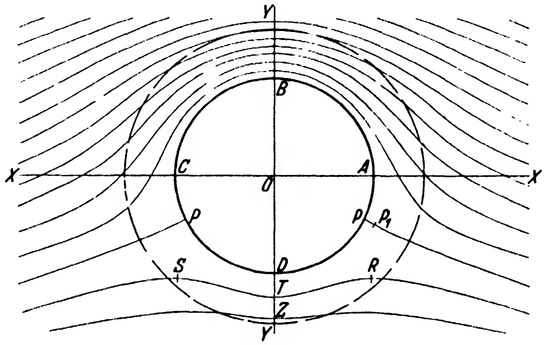

Fig. 31.

$$U\left(r - \frac{a^2}{r}\right)\sin\theta = -k\log\frac{r}{a}$$

or

$$r\sin\theta = y = \frac{-k\log\dfrac{r}{a}}{U\left(1 - \dfrac{a^2}{r^2}\right)} \tag{4.17}$$

from which the line may be readily traced in terms of the coordinates r and y. Where the point of stagnation is on the circle, this will give, in addition to the circle, two branches as shown in the diagram. These branches, of course, meet the circle at the point of stagnation.

We must now find the direction of these branches $\psi = 0$ at the point of stagnation. To this end we put $r = a\,(1 + e)$ in (4.17), e being small. This gives:

$$\sin\theta = \frac{-k\log(1+e)}{Ua\left(1 + e - \dfrac{1}{1+e}\right)}$$

But with e small, $\log(1 + e) = e$ and $1/(1 + e) = 1 - e$. This gives

$$\sin\theta = -\frac{k}{2aU} \quad or \quad y = -\frac{k}{2U}$$

This is, in fact, the value of $\sin\theta$ for a point P_1 just outside of P. But where e is small, this value is the same as for P. This shows that the line $\psi = 0$ approaches and meets the circle in the direction of the radius to the point of contact and hence in a direction $\perp$ to the circle at the same point.

With a value $k = 0$, the point of stagnation will be at the point where the axis of X meets the circle (as in 2) and as k increases from zero, the point will drop down the lower semi-circumference of the circle in accordance with (4.16). To find what value of k will bring this point down to D, we put $\sin\theta = -1$ giving

$$k = 2aU$$

We have next to inquire as to what becomes of the point of stagnation when $k > 2aU$. Equation (4.11) shows that the radial velocity u is always zero on the axis of Y. For a zero value of the velocity v, (4.12) gives the condition

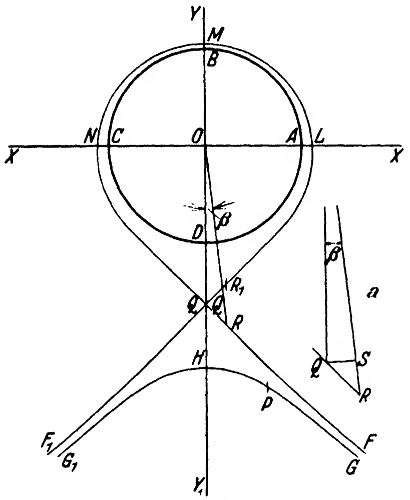

Fig. 32.

$$k = -rU\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

and for points on the axis of $-Y$, this becomes

$$k = rU\left(1 + \frac{a^2}{r^2}\right) \qquad (4.18)$$

Now assume a point on Y beyond D, that is where $r > a$. It is then readily shown that this will require a value of $k > 2aU$. Whence *vice versa* we conclude that for $k > 2aU$, there will be a point of stagnation on YY_1 as given by the value of r from (4.18).

We have thus, for such a point of stagnation (Q Fig. 32) the values of k and r. If then we insert these values in (4.7) with $\sin\theta = -1$, we shall find the value of ψ for the point Q. This value put for ψ in the general equation (4.7) will then give the equation to the particular stream-line, with point of stagnation at Q. A rough tracing of the curve by means of a few points will give a path something like $FQLMNQF_1$. It should be understood that the two parts of the line do not cross at Q. A particle following this path must be understood as coming along FQ, gradually approaching and stopping at Q, then turning and passing along $QLMN$ back to Q where it stops again, then turns and passes along QF_1 indefinitely toward infinity on the down stream side.

A special point of interest here is the angle at which these lines approach and meet the axis of Y and each other. We shall indicate in outline the method by which this question may be examined.

At the point Q, Fig. 32a the general equation (4.7), with the proper value of ψ, is fulfilled. Denote the value OQ by r_1. Now suppose a value of r taken equal to $r_1(1 + e)$ where e is small. Let R be the point on the stream-line corresponding to this value of r. Then $OR = r$. Denote

the angle QOR by β, counted here for convenience as positive. Then counting θ also as positive we have $\beta = (90^{0} - \theta)$. Let QS be an arc with r_1 as radius. Then the displacement of R relative to Q has the two coordinates QS and SR where $QS = r_1\beta$ and $SR = r_1 e$.

We have now to compare the relative magnitudes of QS and SR at the vanishing limit of e and β.

To this end put the general stream-line equation (4.7) in the form

$$- U \sin \theta \left(r - \frac{a^2}{r}\right) = \psi + k \log \frac{r}{a}$$

Remembering our change of sign in counting β, we substitute as follows:

For $- \sin \theta$, $\cos \beta$

For r, $r_1 (1 + e)$

Then with e small we may write

$$\cos \beta = \sqrt{1 - \sin^2 \beta} = \sqrt{1 - \beta^2} = 1 - \frac{\beta^2}{2}$$

$$\frac{1}{1 - e} = 1 - e + e^2 - \ldots\ldots$$

We next write the same equation for the point Q with $\sin \theta = 1$ and $r = r_1$ and subtract this equation from the preceding. The remainder will be an equation involving e and β with known quantities. Among the terms involving e will be found $k \log (1 + e)$. With e small this becomes $k (e - e^2/2)$ (Expansion by McLaurin's Theorem). Putting in this value, remembering that at Q we have $U (r + a^2/r) = k$ (4.18), reducing algebraically and solving for β^2 we have:

$$\beta^2 = \frac{e^2 \left(k - \frac{2 U a^2}{r_1}\right)}{U \left(r_1 - \frac{a^2}{r_1}\right) + k e - \frac{a^2 e^2 U}{r_1}}$$

In the denominator, the terms in e and e^2 will disappear at the limit in comparison with the other terms, and we shall have remaining

$$\beta^2 = e^2 \frac{\left(k - \frac{2 U a^2}{r_1}\right)}{U \left(r_1 - \frac{a^2}{r_1}\right)}$$

If then we substitute back the value of k at Q from (4.18) and reduce, we find finally $\beta = e$

whence $r_1 \beta = r_1 e$ or $QS = SR$

Hence the line FRQ must approach the axis of Y at an angle of 45^{0}.

A similar result will be found for R_1 by taking $r = r_1 (1 - e)$.

Hence the two lines RQ and $R_1 Q$ just meet at an angle of 90^{0}. The four lines at Q form, therefore, four right angles around which the nearby stream-lines must flow.

It thus appears that the fluid above FR and near by will sweep along into and around the angle at Q, thence flowing up and around the line $QLMN$ and so into the angle at Q on the left, then turning and flowing away indefinitely toward ∞ along QF_1 as a boundary. A second part of the fluid will flow similarly along the under side of FQ as a boundary, into the angle at Q and out again along the under side of QF_1 to ∞. A third part of the fluid will remain and circulate indefinitely around the circle within the boundary $QLMNQ$. It is of interest to note the large cross section for flow across DQ due to the very low velocity in this part of the field, as compared with the small section and high velocity across BM.

For points of stagnation on the circle, the value of ψ, as we have seen, is zero. For points of stagnation on the axis of Y below the circle, the value of ψ is not zero and in such case the line $\psi = 0$ will not contain the point of stagnation. The reader will readily satisfy himself that, on the axis of Y, r for the line $\psi = 0$ is greater than r for the point of stagnation and hence the line $\psi = 0$ will lie below the line containing the point of stagnation.

Thus in the diagram, GHG_1 is a line $\psi = 0$ lying below the line containing Q. The values of the velocities c and n show that at all points on the axis of Y (other than Q) c is the only velocity and hence the direction of flow must be parallel to the axis of X. Hence the line GHG_1, as well as all other stream-lines (that containing Q alone excepted), will cross Y in the direction of the axis of X.

Regarding the line for $\psi = 0$, it should be remembered, from the fundamental interpretation of ψ in composite fields (see IV 1) that the significance of the line GHG_1 is that, at any point on this line, such as P for example, the total flow in the indefinite field with the circle, between the datum for that flow (the axis of X and the circle) and the stream-line of that flow passing through P, will be equal in amount and opposite in sign (according to the conventions employed) to the total flow in the vortex field between the datum for that flow (the circle) and the stream-line of that flow passing through P. With this convention of sign then, the total flow is algebraically zero and the actual direction of movement (tangent to the stream-line) is determined by the combination of the two component velocities at that point.

The general character of the field with the point of stagnation on the axis of Y below D, as in Fig. 32, will be readily seen from the indications of this diagram.

5. Pressure on a Circular Boundary in the Field of 4. The total resultant pressure on a circular boundary described about O Fig. 33 as center is a matter of special import in connection with the study of this field of flow. In a perfect fluid, the total pressure at any point is equal in all directions and hence on an element of a surface must be taken

as $\perp$ to the surface at that point. We have, therefore, to find a general expression for the pressure at any point on the circumference of a circle of radius r, and then take its components along X and Y. The sum of such components for the entire circumference will then give the total resultant X and Y forces acting over such boundary.

Bernoulli's equation gives in general

$$p_0 + \frac{1}{2}\varrho U^2 = p + \frac{1}{2}\varrho V^2$$

or

$$p = p_0 + \frac{1}{2}\varrho U^2 - \frac{1}{2}\varrho V^2$$

But p_0 and $(1/2)\varrho U^2$ are both constant and it is clear that a component of a constant pressure taken in any fixed direction and summed around any closed contour will be zero. The problem of the resultant of such pressures is the same as that of the equilibrium of a thin flat disc of any contour whatever, submerged horizontally in a liquid. The horizontal pressures on such a disc are in equilibrium, regardless of the shape of the contour, and hence will be so for a circle. This result is furthermore shown readily by a simple integration around the circle.

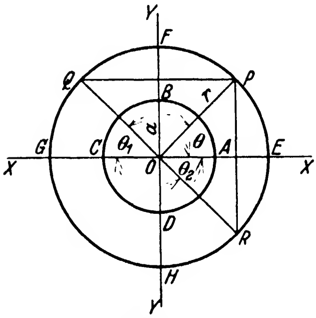

Fig. 33.

With p_0 and $(1/2)\varrho U^2$ disposed of, there remains only V^2 to be considered. At any point P the pressure with which we have to deal is therefore $-(1/2)\varrho V^2$. This pressure over an element $r d\theta$ of the circumference is $-(1/2)\varrho r V^2 d\theta$ and the horizontal component of this is

$$d p_x = -\frac{1}{2}\varrho r V^2 \cos\theta\, d\theta \tag{5.1}$$

Careful attention must here be given to the matter of direction and note may be made at this point of the two different meanings which may attach to the signs plus and minus. In many cases they imply *magnitude* in the sense of one side or the other of zero. In other cases they imply simply direction, as up or down, east or west, in or out. With regard to fluid pressures, so far as magnitude is concerned we cannot have a negative pressure, that is, a tension. The fluid will break. On the other hand we may have fluid pressures acting in any direction, (+) or (—), according to convention. A negative sign attached to a fluid pressure implies therefore direction and not magnitude.

In the present case, if we consider pressure acting from without upon the boundary as positive, that is, from P toward O, then negative pressure will imply force acting in the direction OP. That is, the term $-(1/2)\varrho V^2$ may be considered as representing a force acting along the

direction OP. Such a force, throughout the first quadrant will have a component along $+X$. This will be properly indicated by reversing the sign above and writing

$$d\,p_x = \frac{1}{2}\,\varrho\, r\, V^2 \cos\theta\, d\,\theta \tag{5.2}$$

Such changes in sign can always be made, as desired, since they only imply a matter of convention as to direction. We may also for convenience, omit the constant factor $\varrho r/2$ and insert at a later point as may be required. We have then to consider the expression [see (4.13)].

$$\left.\begin{aligned} V^2\cos\theta = {} & U^2\left(1-\frac{a^2}{r^2}\right)^2\cos^3\theta + U^2\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta\cos\theta + \\ & + \frac{2\,k\,U}{r}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta + \frac{k^2}{r^2}\cos\theta \end{aligned}\right\} \tag{5.3}$$

For every point P this expression will have a value with a positive sign. For every point Q such that $\theta_1 = \theta$, we shall have the same numerical value but with negative sign due to change in the sign of $\cos\theta$ in the second quadrant. These elements will mutually balance and hence the integration for the half circumference EFG will be zero.

Exactly the same result will hold for the third and fourth quadrant, and hence for the circumference as a whole, the X component of the total pressure over such a circular boundary will be zero. That is, there will be no net resultant force along the X direction acting over such a boundary.

For the Y component we have similarly

$$\left.\begin{aligned} V^2\sin\theta = {} & U^2\left(1-\frac{a^2}{r^2}\right)^2\sin\theta\cos^2\theta + U^2\left(1+\frac{a^2}{r^2}\right)^2\sin^3\theta + \\ & + \frac{2\,k\,U}{r}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta + \frac{k^2}{r^2}\sin\theta \end{aligned}\right\} \tag{5.4}$$

Here again the term $-(1/2)\,\varrho\, V^2$ implying a force acting along OP, will have a positive or upward component in the first quadrant. This will be properly indicated by taking the $V^2\sin\theta$ in (5.4) with the $(+)$ sign. Again comparing the values in this equation for two points such as P and R where $\theta_2 = -\theta$, it will be clear that any term containing $\sin\theta$ or $\sin^3\theta$ with other terms which have the same sign in the first and fourth quadrants, will have, for such points, equal numerical values with opposite algebraic signs and hence such terms will cancel out in the summation. It results that only the term in $\sin^2\theta$ will remain and this integrated gives as follows:

$$p_y = \frac{\varrho}{2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} V^2\sin\theta\, r\, d\,\theta = \varrho\, k\, U\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta\, d\,\theta = \frac{\pi\,\varrho\, k\, U}{2}\left(1+\frac{a^2}{r^2}\right)$$

This is for one-half the cylinder. Similar conditions will exist in the second and third quadrants and for the whole cylinder we shall have:

$$p_y = \pi \varrho k U \left(1 + \frac{a^2}{r^2}\right) \tag{5.5}$$

or putting in the value of $k = \Gamma/2\pi$

$$p_y = \frac{\varrho \Gamma U}{2} \left(1 + \frac{a^2}{r^2}\right) \tag{5.6}$$

These results are positive in sign and therefore imply a resultant force upward; that is, a lift force over the boundary of such a circle measured by the value in (5.6).

If we consider what this becomes on the limit circle of radius a we have:

$$p_y = \varrho \Gamma U \tag{5.7}$$

But, as we have seen, this circle may represent the boundary of a solid body of circular cross section placed in a stream of combined translation and vortex flow and we thus have as a final result the conclusion that the total lift on a cylinder of indefinite length placed in a field of flow composed, as here assumed, of translation and circulation, will experience, per unit length, a total force at right angles to the stream, measured by the expression in (5.7).

6. Change of Momentum within any Circular Boundary in the Field of 4. Another item of importance in the study of this field is the change of momentum within any circular boundary with radius r and with O as center. The change of momentum within such a boundary will obviously be equal to the net flow of momentum across the boundary. If at any instant the flow inward and outward are equal there is no change within the boundary. If they are not equal, there is a change measured by such a net flow. Note the stream-line flow in Fig. 31.

In Fig. 33, let P be a point in the first quadrant. The radial velocity at this point is n and the length of an element of arc is $r d\theta$. Let Q denote volume of flow. Then for this arc we shall have

$$d Q = n r d \theta$$

This entering fluid will have an X component velocity u and a Y component velocity v. The element of X momentum crossing $r d\theta$ will therefore have for its numerical measure

$$d M_x = \varrho u d Q = \varrho u n r d \theta$$

and similarly for the element of Y momentum

$$d M_y = \varrho v d Q = \varrho v n r d \theta$$

With regard to sign, we will take flow crossing $r d\theta$ inward as positive, and outward as negative. In the first quadrant therefore, $d Q$ is $(+)$ while u is $(-)$ and v is $+$. For the X momentum, we therefore take the element as $(-)$ and for the Y momentum as $(+)$. But in this

quadrant u and n are both (—) and the product will be + and we must therefore put a minus sign before the product $u n$ and write

$$dM_x = -\varrho\, u\, n\, r\, d\,\theta \tag{6.1}$$

In a similar manner we note that v is (+), while n is (—) and the product is (—). Hence for $d M_y$, we must likewise give to the product $v\, n$ the (—) sign and write

$$d M_y = -\varrho\, v\, n\, r\, d\,\theta \tag{6.2}$$

Putting in the values of u and n we have as follows, omitting, for convenience, the constants ϱ and r:

$$\left.\begin{aligned} u\, n\, d\,\theta = U^2\left(1-\frac{a^2}{r^2}\right)\cos\theta\, d\,\theta - U^2\frac{a^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\cos 2\,\theta\cos\theta\, d\,\theta\, + \\ +\frac{k\,U}{r}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta\, d\,\theta \end{aligned}\right\} \tag{6.3}$$

Now considering points like P and Q where $\theta_1 = \theta$, it is clear that as with the case of pressures, any term in $\cos\theta$ or $\cos^3\theta$ and with other factors the same in both quadrants will give, for such points, the same numerical values but with opposite signs, and that such pairs of values will cancel out in the summation. But this includes all of the terms of the product $u n$. The same will be true for pairs of points in the third and fourth quadrants. It thus results that for the entire circumference the net flux of X momentum across the boundary is zero. That is, the rate of flux inward is equal to the rate of flux outward and the result is no change.

Again putting the value of v and n in (6.2) we have

$$-v\, n\, d\,\theta = \frac{2\,a^2\,U^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos^2\theta\, d\,\theta + \frac{k\,U}{r}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta\, d\,\theta \tag{6.4}$$

Here again for pairs of points such as P and R, where $\theta_2 = -\theta$, the term in $\sin\theta\cos^2\theta$ will give values equal numerically but opposite in sign and thus cancelling out in the summation. This leaves only the term in $\cos^2\theta$ and we have (putting in ϱ and r)

$$d\,M_y = \varrho\, k\, U\left(1-\frac{a^2}{r^2}\right)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^2\theta\, d\,\theta$$

or

$$\triangle\, M_y = \frac{\pi\,\varrho\, k\, U}{2}\left(1-\frac{a^2}{r^2}\right) \tag{6.5}$$

This is for one-half of the cylinder and gives the measure of the rate of *inflow* of *upward* momentum over this boundary.

Now both the equations and the symmetry of the stream-line system about $Y\,Y$ show that in the second and third quadrants the conditions must be exactly the same in quantitative measure but reversed in sign. Hence over this half of the boundary there will be the same expression.

The total rate of change of momentum within the boundary will therefore be measured by the change of the momentum in (6.5) from up to down—that is, by its complete reversal. This is the same as stopping it in the up direction and starting it in the down. The total change will be, therefore, twice this expression or, for the entire surface

$$\triangle M_y = \pi \varrho k U \left(1 - \frac{a^2}{r^2}\right)$$

Or otherwise putting in $\Gamma/2\pi$ instead of k we have for the entire surface:

$$\triangle M_y = \frac{\varrho \Gamma U}{2} \left(1 - \frac{a^2}{r^2}\right) \tag{6.6}$$

Consider now the equilibrium of the mass of fluid between the cylinder of radius a and the boundary of any outlying cylinder of radius r (see Fig. 33). The fluid within this cylindrical shell can be acted on, so far as external forces are concerned, only by the pressures on its inner and outer boundaries, that is, by the pressure acting over the surface of $ABCD$ and directed from the latter to the fluid, combined with the pressure acting over the outer surface $EFGH$ from the outside upon the fluid. Now whatever the resultant of such forces may be, they alone can be responsible for whatever changes in momentum are occurring within this cylindrical shell. Or otherwise, we know that a rate of change of momentum is evidence of the operation of a force and the measure of the rate of change will be a measure of the force producing it.

Now we have seen that there is a change of momentum continually impressed upon the fluid within the outer boundary of radius r, that the rate of such change is as given in (6.6) and that its direction is *downward*. Hence there must be a net downward force acting on this mass of fluid measured by this same expression. But we have seen that there is a net upward pressure over the outer boundary measured as in (5.6). Hence the downward pressure must come from the cylinder $ABCD$ and the difference between such pressure down and the pressure on the outer boundary up must furnish the final resultant net downward force equal to the rate of change of momentum as measured in (6.6).

Let P_C = downward force exerted by cylinder on fluid.

P_B = upward force exerted by outer fluid over boundary $EFGH$.

$\triangle M$ = rate of change of momentum downward.

Then we must have $$P_C - P_B = \triangle M$$

or $$P_C = P_B + \triangle M$$

Then putting in the values of P_B and $\triangle M$ from (5.6) and (6.6) we have finally

$$P_C = \varrho \Gamma U \tag{6.7}$$

This is the same expression as in (5.7) and implies simply the equality of action and reaction. If the fluid presses upward on the cylinder

with a force of $\varrho \Gamma U$, then the cylinder must press downward on the field with the same force $\varrho \Gamma U$.

7. Total Resultant Force on any Body in Field of 4. From (4.9), (4.10) for the combination of 4 we have for the u and v components of the velocity,

$$u = -U + \frac{Ua^2}{r^2} \cos 2\theta - \frac{k}{r} \sin \theta \tag{7.1}$$

$$v = U\frac{a^2}{r^2} \sin 2\theta + \frac{k}{r} \cos \theta \tag{7.2}$$

In these equations, it will be seen, by reference to II 4, IV (10.4) and III 3 that in the value for u, the first term represents the rectilinear flow, the second represents the influence of the doublet and the third that of the vortex or circulation flow. Similarly for v, the first term represents the doublet and the second the vortex flow.

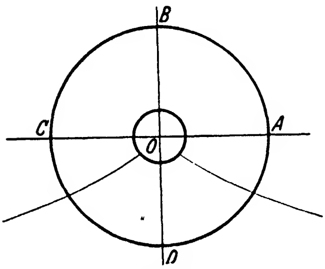

Fig. 34.

Let Fig. 34 denote the field of flow with cylinder at O. Then a consideration of these various terms in (7.1) (7.2) shows that when r is large the influence of the doublet or cylinder becomes negligible and the field becomes simply that due to the flow $-U$ combined with the vortex.

Thus in Fig. 34 if $ACBD$ is a circle of radius very large relative to the size of the cylinder, then over the circumference of $ABCD$ or over a cylindrical surface of radius OA, the conditions of flow will be sensibly those pertaining only to the flow $-U$ combined with the vortex movement as would result in (7.1) (7.2) by the omission of the terms with r^2 in the denominator. The same will remain true if the cylinder at O is removed entirely, or if any other body of similar dimensions is placed in its stead. In other words, the flow over or through a cylindrical surface of radius OA which is very large will not be sensibly affected by the *shape* of a small body placed at O.

Or expressed otherwise, with such a flow around a small body of any shape whatever placed at O, we can always go far enough away to find a cylindrical surface at which the flow will not be sensibly affected by the shape of the body and which will therefore be that for the flow $-U$ with vortex movement about O.

Now let us apply the momentum theorem (I 7) to the fluid lying between the cylinder $ABCD$ and the body at O.

Let $\triangle M$ = Rate of Change of Momentum.

F_C = Force resulting from pressure over cylinder $ABCD$.

F_B = Force resulting from pressure over surface of body.

But from 5, 6 we know that F_C is directed upward and $\triangle M$ downward. Hence in the downward sense,

$$F_B - F_C = \text{net force measured by } \triangle M$$

or

$$F_B = F_C + \triangle M$$

Likewise we know that

$$F_C = \frac{1}{2} \varrho \Gamma U \left(1 + \frac{a^2}{r^2}\right)$$

$$\triangle M = \frac{1}{2} \varrho \Gamma U \left(1 - \frac{a^2}{r^2}\right)$$

Whence

$$F_B = \varrho \Gamma U \tag{7.3}$$

But the force acting from the fluid on the body will be F_B reversed in direction, and we shall therefore have for this force or lift the same measure, $\varrho \Gamma U$.

We thus reach the important conclusion that the resultant Y force acting on a body of uniform cross-sectional form placed in a field of flow composed of a velocity of translation $-U$ with vortex motion (circulation) about the cross section, will be independent of the form of the cross section of the body and will be measured by $\varrho \Gamma U$. This result, usually known under the name of the Kutta-Joukowski theorem, has many important applications in connection with aeronautic problems.

In the same manner and resulting from the same general course of reasoning, we derive the conclusion that the resultant X force on a body of indefinite length in a field of flow of this character is zero. This again, of course, is not in accord with the facts for actual fluids, all of which is a further illustration of the fact that the assumption of a zero or of a vanishing viscosity, while giving many results of great interest and closely approximate to the facts of experience with actual fluids, may yet in other matters lead to widely erroneous results.

CHAPTER VI

APPLICATION OF CONFORMAL TRANSFORMATION TO FIELDS OF FLOW

1. The Application of Conformal Transformation to the Study of Fields of Fluid Motion. A field of fluid motion, as we have seen, is characterized by two functions φ and ψ, or by their combination into the potential function w. Furthermore, either of the functions φ or ψ may be taken as a stream function, and whichever is so taken, the other will then serve as the velocity potential, Unless otherwise specified, however, it is customary to take φ for the velocity potential and ψ for the stream function.

The mathematical conditions for the existence of such a field with its two functions φ and ψ are expressed in Division A VII (4.1), (4.2) the first of which expresses the conditions for irrotational motion and the second for flow as an incompressible fluid. Now suppose such a field subjected

to *conformal transformation* and thus changed entirely in its geometrical form. Will the field thus transformed meet the conditions for a field of fluid motion? Will the necessary mathematical conditions, as above, be fulfilled in the new field as well as in the old? This question is most readily answered by a direct examination of the physical conditions in the transformed field as derived from those of the original.

In Fig. 35 let $A_1 B_1 C_1 D_1$ be any small element of field A lying between two stream-lines ψ_1 and ψ_2 and twoe quipotential lines φ_1 and φ_2. Let $A_2 B_2 C_2 D_2$ represent this same element as transformed into field B.

Now in field A we know that two fundamental conditions are fulfilled:

(1) The flow is irrotational—line integral about any closed area in the field is zero.

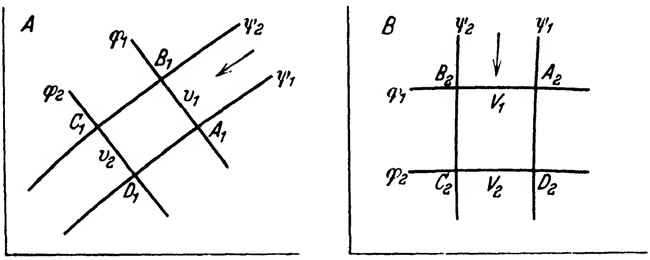

Fig. 35.

(2) The flow is that of an incompressible fluid—equation of continuity satisfied.

In the two diagrams of Fig. 35, it is to be understood that, as representing fields of flow, ψ_1 and ψ_2, φ_1 and φ_2 represent, in both fields, the same values of ψ and φ. In fact they must be viewed as representing the same flow, with simply a change in the geometry of the two fields. It will follow that the combination of the two functions φ and ψ in the potential function w will likewise, at corresponding points, have the same value, and we therefore use the same notation φ, ψ and w, implying the same values in either field, parent or transformed.

From the conditions of conformal transformation, we know that the two elements $A_1 B_1 C_1 D_1$ and $A_2 B_2 C_2 D_2$ will be similar and proportional in all respects and that the linear ratio at corresponding points in the two fields is furnished by the scalar value of $f'(z)$ for such points.

From the above it follows that the flow $(\psi_2 - \psi_1)$ will be the same in the two fields and hence, denoting the velocities as in the diagram, we shall have

$$v_1 A_1 B_1 = V_1 A_2 B_2$$
$$v_2 C_1 D_1 = V_2 C_2 D_2$$
$$\text{etc.} \qquad \text{etc.}$$

or

$$\frac{V_1}{v_1} = \frac{A_1 B_1}{A_2 B_2}$$

In words, the velocities at corresponding points in the two fields are in the inverse ratio of the linear dimensions and hence inversely proportional to the ratio of linear transformation.

Again if φ_1 and φ_2 have the same value in the two fields, $(\varphi_2 - \varphi_1)$ will be the same. But this is the line integral between φ_1 and φ_2. Hence, denoting line integral by L, we shall have $L_{A1\,D1} = L_{A2\,D2}$. Then since A_2D_2 and A_1D_1 are in the direct ratio of linear transformation, it follows that the velocities along A_2D_2 and A_1D_1 will be in the inverse ratio of linear transformation. Thus either way we reach the same result regarding the velocities at corresponding points in the two fields and in general it appears that at such points the geometrical conditions are in all respects directly similar and the velocity relations are inversely similar, in the same ratio.

Again in the above equation, since $v_1A_1B_1 = v_2C_1D_1$, we must have

$$V_1A_2B_2 = V_2C_2D_2$$

with, of course, no flow across A_2D_2 and B_2C_2. That is, the fluid in field B will flow in conformity with the law of continuity, *i. e.*, as an incompressible medium.

The same geometrical and velocity relations serve to show that the conditions for irrotational flow are also fulfilled. The element $A_1B_1C_1D_1$ is, of course, at the limit a rectangle with zero velocity along A_1B_1 and C_1D_1 and with the relation

$$L_{A1\,D1} = L_{B1\,C1} = -L_{C1\,B1}$$

and hence $$L_{A1\,D1} + L_{C1\,B1} = 0$$

But from the geometrical relations between the two fields, $A_2B_2C_2D_2$ will likewise be a small rectangle, there will be no velocity in the directions A_2B_2 and C_2D_2 and by the same reasoning as for field A, we shall have

$$L_{A2\,D2} + L_{C2\,B2} = 0$$

or line integral around a small element in field B is 0 and thus the conditions for irrotational flow are fulfilled.

We conclude, therefore, that any known field of fluid flow fulfilling the conditions as contemplated in I 4 and II 1 and mapped completely with stream and equipotential lines, may be transformed complete by conformal transformation into a second field of flow, meeting all the physical requirements for flow of the same character as in the original, and such that the numerical relations between the two fields at corresponding points will be directly determined by the transformation ratio at that particular point.

2. Velocity Relations between Fields of Flow on the z and Z Planes. The relation $dZ = f'(z)\,dz$ may often be employed with advantage in the comparison of the characteristics and properties of two fields of flow, one as derived from the other by conformal transformation. This

equation gives a definite relation between the length of any small linear element dz on the z plane and that of the corresponding element dZ on the Z plane.

Thus at any point P on the z plane, we have for the total velocity of flow, $V = d\varphi/ds$, where s is the direction along the stream-line $\psi = const.$

But ds, as an element of the stream-line, has both length and direction and as such will be represented by dz, an elemental vector on the z plane; and this is transformed to the Z plane through multiplication by the vector $f'(z)$.

Let the scalar value of $f'(z)$ be r and its vector angle θ. Then in the transformation, ds will be multiplied by r and turned through an angle θ and thus transformed it becomes the vector dZ, the corresponding element of ψ on the Z plane.

But in the transformation, $d\varphi$ remains the same and in a scalar sense we shall have for the velocity on the Z plane,

$$V' = \frac{d\varphi}{r\,ds} = \frac{V}{r}$$

Again it is remembered that any given velocity component may be found by taking the partial derivative of φ along the direction desired or that of ψ along a direction 90^0 ahead of that desired (see Division A VIII 4). In the present case, therefore, we shall have equally well, $V = d\psi/ds_1$, where s_1 is the direction along the line $\varphi = const.$ Likewise everything about the field geometrically, close about the point P, will be transformed with the same vector ratio or operator $f'(z)$. We may, therefore, write equally well, $V' = \dfrac{d\psi}{r\,ds_1} = \dfrac{V}{r}$

It follows, therefore, that the total velocities at corresponding points on the two planes will be in the inverse ratio of the scalar value of $f'(z)$, and such total velocity on the z plane may be found either as $d\varphi/ds$ or $d\psi/ds_1$, where s and s_1 imply directions respectively along the lines $\varphi = const.$ and $\psi = const.$

But the elements ds and ds_1 have direction as well as length and as such they are to be denoted by dz. Now let α be the vector angle of $f'(z)$ and θ that of the small element dz on the stream-line $\psi = const.$ Then the vector angle of the corresponding dZ will be $\alpha + \theta$ and this will determine the direction of the flow on the Z plane. But the angle α is determined by expanding $f'(z)$ in terms of x and y, and θ is determined from the relation $\tan\theta = v/u$ where u and v are the x and y component velocities on the z plane. We have thus the complete determination of the velocity on the Z plane in both direction and magnitude, and thence the component velocities along X and Y, or in any other direction as desired, may be readily found.

3. Conformal Transformation of the Circle. An important example of conformal transformation is found in the application of the equation:

$$Z = z + \frac{b^2}{z} \tag{3.1}$$

to a circle of radius a, where b may be equal to or slightly less than a.

Transformation into a Straight Line. In this case $b = a$ and the origin is taken at the center of the circle. The equation to the circle may be written $z = ae^{i\theta}$ [see Division A V 11 (e)] and this will give:

$$Z = z + \frac{a^2}{z} = a\,(e^{i\theta} + e^{-i\theta})$$

and from Division A I 5 (d) this becomes:

$$Z = 2a \cos \theta$$

This is wholly scalar and for successive values of θ will give the successive points on a line lying on the axis of X and extending from A to B, a distance of $2a$ on either side of the center. See Fig. 36. The reason for this is readily seen from the diagram. Let OP represent any value of z. Then since its scalar length is a, the expression a^2/z will be represented by a length $OQ = a$, laid off at $-\theta$. Z is then the vector sum of OP and OQ and this will equal OR. In this way it is seen that the point R will always lie on the line AB, and furthermore, that in going the entire way around the circle, the line will be traversed twice, once from B to A and once from A to B. The result is as though the circle were collapsed to the width of a single line and stretched out to extend the distance between A and B.

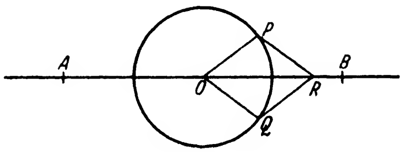

Fig. 36.

Transformation into A Circular Arc. In Fig. 37 the origin of transformation O is taken directly below Q and b is taken equal to OB. An auxiliary circle about O as center and with b as radius will then pass through A and B. The vector z for any point P may be put in the form $z = re^{i\theta}$. Putting this in (3.1) we have

$$Z = X + iY = re^{i\theta} + \frac{b^2}{r} e^{-i\theta}$$

Using the transformation of Division A I 5 (c), (b) we then find

$$X = \left(r + \frac{b^2}{r}\right) \cos \theta \tag{3.2}$$

$$Y = \left(r - \frac{b^2}{r}\right) \sin \theta \tag{3.3}$$

Eliminating r from these two equations we find

$$X^2 \sin^2 \theta - Y^2 \cos^2 \theta = 4b^2 \sin^2 \theta \cos^2 \theta \tag{3.4}$$

Again from the diagram, $a = b \sec \beta$

$$OQ = b \tan \beta$$

Then in the triangle OPQ

$$2\,r\,b\,\tan\beta\,\sin\theta = r^2 + b^2\tan^2\beta - b^2\sec^2\beta$$
$$= r^2 - b^2 \qquad (3.5)$$

Dividing (3.3) by (3.5) we have

$$\sin^2\theta = Y/2\,b\,\tan\beta$$

whence
$$\cos^2\theta = (2\,b\,\tan\beta - Y)/2\,b\,\tan\beta$$

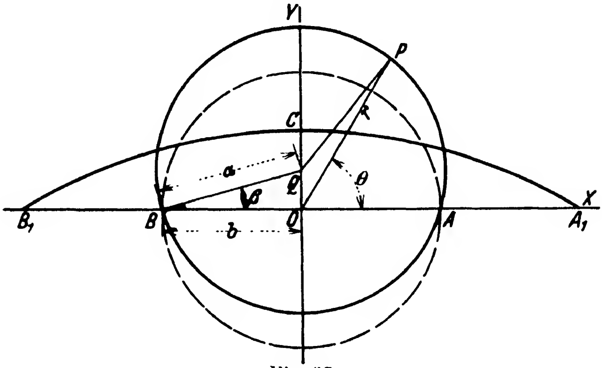

Fig. 37.

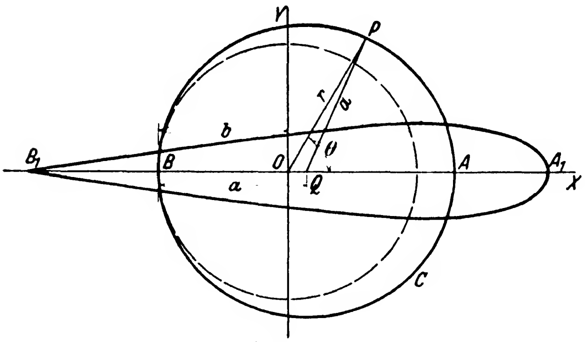

Fig. 38.

Putting these values in (3.4) and reducing we find

$$X^2 + (Y + 2\,b\,\cot 2\,\beta)^2 = (2\,b\,\mathrm{cosec}\,2\,\beta)^2$$

This is a circle of radius $2b\,\mathrm{cosec}\,2\beta$ and center at a distance $2b\,\cot 2\beta$ below O. Hence the height of the arc OC will be $2b\,(\mathrm{cosec}\,2\beta - \cot 2\beta)$ and this readily reduces to $2\,b\,\tan\beta = 2\,OQ$.

Likewise for $Y = 0$, we have

$$X^2 = 4\,b^2\,(\mathrm{cosec}^2\,2\,\beta - \cot^2 2\,\beta)$$

or
$$X = \pm 2\,b$$

and
$$OA_1 = OB_1 = 2\,b.$$

Transformation into Symmetrical Forms. In Fig. 38 the origin O is taken on the diameter AB and b is taken equal to OB. The auxiliary

circle with radius b will then be as shown in the diagram. If OQ is small compared with the other dimensions, the resulting form may be developed to a close approximation as follows:

Put $a = b(1 + e)$ where e is small. Then for the point A we have $z = b(1 + 2e)$ and

$$Z = b(1 + 2e) + \frac{b^2}{b(1 + 2e)}$$

Hence $\quad OA_1 = b(1 + 2e + 1 - 2e + 4e^2 \ldots)$ app.

or $\quad OA_1 = 2b(1 + 2e^2)$ app.

Also $\quad OB_1 = 2b$

Hence $\quad A_1B_1 = 4b(1 + e^2)$

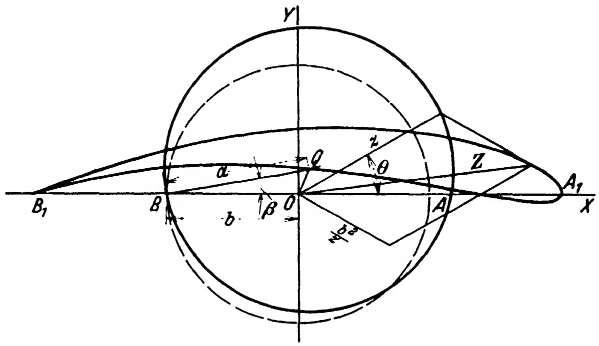

Fig. 39.

For most purposes it will be sufficiently accurate to take

$$A_1B_1 = 4b$$

Again in the triangle OPQ

$$a^2 = r^2 + (a - b)^2 - 2r(a - b)\cos\theta$$

Putting in the value of a in terms of b, retaining only terms involving the first power of e and reducing we find

$$r = b[1 + e(1 + \cos\theta)]$$

Whence from (3.2), (3.3)

$$X = \left(r + \frac{b^2}{r}\right)\cos\theta = 2b\cos\theta$$

$$Y = \left(r - \frac{b^2}{r}\right)\sin\theta = 2be(1 + \cos\theta)\sin\theta$$

and from these the form may be constructed point by point.

For $\theta = 90^\circ$, $Y = 2be$ and $Y_{max.}$ is found at $\cos\theta = 1/2$. This gives $Y_{max.} = be\,3\sqrt{3}/2$ at $X = b$. Whence the thickness of a foil of this form will be $4be$ at the center and $3be\sqrt{3}$ at the point of maximum thickness, $X = b$.

Transformation Into an Airfoil Section. In this case the origin O is taken obliquely off center as shown in Fig. 39 and b is here taken equal to OB. The result of the transformation will then be a form representing a close approximation to an airfoil section as shown in the diagram.

4. Transformation of the Flow Along the Axis of X into the Flow about a Circle. We write the transformation of 3 in the form

$$Z = X + iY = z + \frac{a^2}{z} \tag{4.1}$$

and putting in $z = x + iy$, find,

$$X = x\left(1 + \frac{a^2}{r^2}\right) \tag{4.2}$$

$$Y = y\left(1 - \frac{a^2}{r^2}\right) \tag{4.3}$$

where as usual, $r^2 = x^2 + y^2$. Equations (4.2) and (4.3) thus give the coordinate relations between the two planes of z and Z, and represent (4.1) in more extended form. In general then, as we have seen, any line in the plane z will transform into a line in the plane Z, the equation to which in X and Y may be found by eliminating x and y between (4.2) and (4.3) combined with the equation to the line in the plane z.

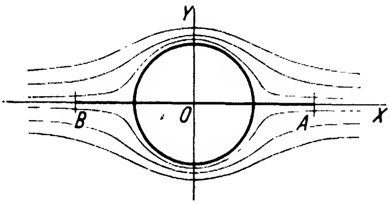

Fig. 40.

Now suppose, in Fig. 40, that we have a circle with the field of flow about it, as in V 2; and suppose the transformation of (4.1) applied to this entire geometrical field. The circle will transform into a straight line AB and from 1 the remainder of the field must transform into the flow about AB. But we already know this as a field of straight lines parallel to X, and with a uniform velocity which we may take as $-U$ (in the direction of $-X$). If we then derive the stream and velocity potential functions for such a flow in the Z field (see II 4) we shall have

$$\left.\begin{aligned} \psi &= -UY \\ \varphi &= -UX \end{aligned}\right\} \tag{4.4}$$

The equation to a stream-line in the Z plane will then be, $Y = const.$, and similarly that to an equipotential, $X = const.$

Now, with all this in mind, suppose that we should start with the Z plane and transform back to the z plane. That is, we start with the straight line AB and the accompanying field of flow, and transform in the inverse direction through (4.1) from Z back to z. We know that AB will transform into the circle, and, as we have seen in 1, the lines of flow on the plane Z must transform into the lines of flow about the circle.

But $Y = const.$ is a stream-line on the Z plane and $Y = const.$ on the Z plane means that $y\left(1 - \frac{a^2}{r^2}\right)$ must be a constant on the z plane. Hence, from (4.3), (4.4)

$$\psi = -UY = -Uy\left(1 - \frac{a^2}{r^2}\right) = \text{const.} \tag{4.5}$$

must be the general equation to the stream-lines on the z plane, or in other words, to the stream-lines about the circle.

In the same way we may show that, on the z plane, for the equipotential lines,

$$\varphi = -UX = -Ux\left(1 + \frac{a^2}{r^2}\right) = \text{const.} \tag{4.6}$$

Hence for the two functions φ and ψ on the z plane,

$$\left.\begin{aligned}\varphi &= -Ux\left(1 + \frac{a^2}{r^2}\right)\\ \psi &= -Uy\left(1 - \frac{a^2}{r^2}\right)\end{aligned}\right| \tag{4.7}$$

If then in accordance with the methods of Division A VIII 8 we derive the form of the potential function w, having given φ and ψ as above, we shall find $w = -U\left(z + \frac{a^2}{z}\right)$ (4.8)

But on the Z plane,

$$w = \varphi + i\psi = -U(Y + iY) = -UZ \tag{4.9}$$

Hence comparing (4.8) and (4.9) we have

$$Z = z + \frac{a^2}{z}$$

which brings us back to the equation of transformation (4.1).

These results for the flow about a circle are the same as those in V 2 derived in a different manner. The present method may, in fact, be shortened up to the following significant steps:

(1) On the Z plane we have

$$w = \varphi + i\psi = -U(X + iY) = -UZ$$

(2) The equation of transformation is

$$Z = z + \frac{a^2}{z}$$

(3) Hence on the z plane we shall have

$$w = -U\left(z + \frac{a^2}{z}\right)$$

and from which the values of φ and ψ in (4.7) are readily found.

In the present case, we have preferred a more detailed development in order the better to bring out the relations between the two fields z and Z and to illustrate the method of transforming the functions φ, ψ and w from one field to the other.

We have thus two separate and independent methods whereby the flow about a circle may be obtained. (1) By combining a doublet with a rectilinear indefinite flow as in V 2, and (2) By the conformal transformation of the rectilinear flow about a straight line $A\,B$ as in the present section.

5. Transformation of the Flow about a Circle into the Flow about a Straight Line at Right Angles to the Flow. Let us take now the transformation

$$Z = \left(z - \frac{a^2}{z}\right) = X + iY \tag{5.1}$$

and apply to a circle of radius a. Referring again to Division A I 5 (e) this becomes $Z = 2ia\,\sin\theta$.

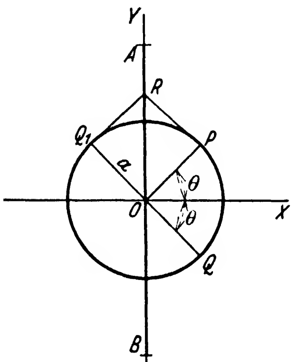

Fig. 41.

This is a distance $2a\,\sin\theta$ laid off on the axis of Y, and in the same manner as in 3 it is seen that the result will be a line $A\,B$, $2a$ in length, as in Fig. 41. This is again readily seen from the diagram. The vector z is represented by OP, the vector a^2/z by OQ and the vector $-a^2/z$ by $O\,Q_1$. The combination of $O\,P$ and $O\,Q_1$ will obviously give the point R on the axis of Y.

Suppose now that we have the circle with its surrounding field of flow as in V 2 and that the transformation (5.1) is applied to the entire field including the circle. As we have just seen, the circle will transform into the line $A\,B$ and the field of flow must then transform into a field of flow along X and around $A\,B$. This should give the field of flow of an indefinite stream about a straight line placed at 90^0 to the direction of flow. But for the field in the plane of z,

$$w = -U\left(z + \frac{a^2}{z}\right) = \varphi + i\psi \quad \text{[see V (2.9)]} \tag{5.2}$$

Combining (5.1) and (5.2) by squaring each equation and subtracting we find

$$w = -U\sqrt{Z^2 + 4a^2} \tag{5.3}$$

The minus sign is taken for $\sqrt{U^2}$ since it was $-U$ which was squared and this retains the convention of a velocity U from left to right.

This gives w as a function of Z, but not in a form readily adapted to computation.

Putting $z = x + iy$ in (5.1) and (5.2) we find

$$X = x\left(1 - \frac{a^2}{r^2}\right) \tag{5.4}$$

$$Y = y\left(1 + \frac{a^2}{r^2}\right) \tag{5.5}$$

$$\varphi = -Ux\left(1 + \frac{a^2}{r^2}\right) \tag{5.6}$$

$$\psi = -Uy\left(1 - \frac{a^2}{r^2}\right) \tag{5.7}$$

For any given point x, y, in the plane z, the values of X and Y are readily found from (5.4) and (5.5) and the point in the plane Z thus determined. For any given stream-line in z, ψ will have a constant value and with this value, y may be taken in (5.7) and corresponding values of r and x found. These may then be put in (5.4) and (5.5) and the point thus located in the Z plane.

By a process of algebraic elimination of x and y between (5.4), (5.5), (5.6), and again between (5.4), (5.5), (5.7), we find equations to the equipotential and to the stream-lines in Z as follows:

$$(U^2 X^2 - \varphi^2)(U^2 Y^2 + \varphi^2) = -4a^2 U^2 \varphi^2$$
$$(U^2 Y^2 - \psi^2)(U^2 X^2 + \psi^2) = 4a^2 U^2 \psi^2$$

From these equations in X and Y the entire field may be mapped and its properties examined. The character of the field is shown in Fig. 42.

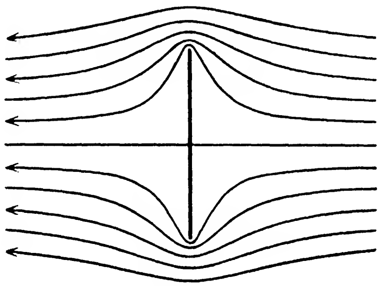

Fig. 42.
(From Tietjens, Hydro- und Aeromechanik, nach Vorlesungen von L. Prandtl, Vol. I. 1929.)

If we suppose the field of flow in Fig. 42 revolved through $+90°$, the results will give the flow in the direction of $-Y$ past a line lying in the axis of X and of length $4a$. To transform the circle into this line we use the transformation of 3

$$Z = z + \frac{a^2}{z}.$$

Combining this with V (2.21) in the same way as for (5.1) and (5.2) above, we have, for the potential function

$$w = iV\sqrt{Z^2 - 4a^2} \tag{5.8}$$

For a line of length $2b$ and with the extended expression for Z this becomes

$$w = iV\sqrt{(X^2 + iY)^2 - b^2} \tag{5.9}$$

By a different mode of treatment, the functions φ and ψ may be expressed in terms of X and Y, the coordinates on the Z plane, thus leading more directly to expressions giving the velocity and pressure distribution in the field. The strategy to be employed in this case is also of interest since it illustrates a general method which is frequently of use in dealing with problems of this character.

From (5.2) (5.3) we have

$$w = \varphi + i\psi = -U\sqrt{Z^2 + 4a^2}$$

Put

$$\left.\begin{aligned} \varphi &= -UA \\ \psi &= -UB \end{aligned}\right\} \tag{5.10}$$

This gives

$$\sqrt{Z^2 + 4a^2} = A + iB$$

As an equation for the determination of A and B in terms of X and Y. To this end we put $X + iY$ for $\boldsymbol{Z}$, square both sides and separate the real and imaginary parts. This will give two equations as follows:

$$A^2 - B^2 = X^2 - Y^2 + 4a^2$$

$$AB = XY$$

These we may treat as simultaneous equations in A and B and solve. This will give:

$$A = \left(\frac{P+Q}{2}\right)^{1/2}$$

$$B = \left(\frac{-P+Q}{2}\right)^{1/2}$$

Where $P = X^2 - Y^2 + 4a^2$ and $Q = \sqrt{4X^2Y^2 + P^2}$

These values of A and B put in (5.10) will then give φ and ψ. For the velocities u and v we have simply to find $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$.

Carrying out the differentiations and reducing we find

$$\left.\begin{aligned} u &= -UX\frac{(P+Q+2Y^2)}{2AQ} \\ v &= UY\frac{(P+Q-2X^2)}{2AQ} \end{aligned}\right\} \qquad (5.11)$$

In the interpretation of the expressions for A and B it will be noted that, from the physical nature of the problem, A and B are always real. Hence the sign of Q must always be so taken as to give real values for A and B. Examination will show that to this end the $(+)$ sign is required. However, P may be either positive or negative but P^2 is, of course, always positive. With these interpretations (5.11) will give the values of u and v for the Z plane according to the values of X and Y and the resulting values of A, P, and Q. It may be noted in this form of solution, that, in squaring, the distinction between the $(+)$ and $(-)$ signs has been lost and in applying the expressions for A and B to the different quadrants of the field, an appropriate choice of the double sign of the $\sqrt{\ }$ must be made. However, the general character of the field, as shown in Fig. 42, will provide a ready guide for this choice.

For points on the line where $X = 0$ and $Y < 2a$, we have P positive $= 4a^2 - Y^2$, $Q = P$ and $A = \sqrt{P}$. Whence from (5.11)

$$\left.\begin{aligned} u &= 0 \\ v &= \frac{UY}{\sqrt{4a^2 - Y^2}} \end{aligned}\right\} \qquad (5.12)$$

For points on the axis of Y beyond $Y = 2a$, that is, in the field beyond the line, we have P negative $= 4a^2 - Y^2$, Q positive as before $= Y^2 - 4a^2$ and $A = 0$. The use of (5.11) in such cases results in indeterminate values and although these may be evaluated, the need of such a step may be avoided by taking $u = \partial\psi/\partial Y$ and $v = -\partial\psi/\partial X$ and thus taking derivatives of B instead of A.

For such case, therefore, with P negative and Q positive, we shall have

$$B = \sqrt{Y^2 - 4a^2}$$

whence

$$\left.\begin{aligned} u &= -U\frac{\partial B}{\partial Y} = \frac{-UY}{\sqrt{Y^2 - 4a^2}} \\ v &= U\frac{\partial B}{\partial X} = 0 \end{aligned}\right\} \qquad (5.13)$$

At the end of the line where $Y = 2a$, (5.12) gives an infinite value for v and (5.11) the same for u. This arises from the fact that the form of the mathematical functions employed, transforms the stream-line which follows the circle $A\,B\,C$ into a stream-line $A_1\,B_1\,C_1$ flowing up one side of the line and down the other and thus making a complete reversal of the direction at the end of the line B. All of this is impossible physically and implies a failure of the velocity values at and near this point.

It may be of interest to illustrate at this point the application of 2 to the determination of the velocity in the Z field. Taking the equation of transformation (5.1) and solving for z we have the reverse form.

$$z = \frac{Z}{2} + \frac{1}{2}\sqrt{Z^2 + 4a^2}$$

Now putting $a = 10$ and $U = -10$, let us take the point in the Z field, $X = 1$, $Y = 21$, or $1 + 21i$. In order to find z, or $x + iy$, we must first find the value of $\sqrt{Z^2 + 4a^2}$ as a vector. To this end we may utilize (5.10) as the solution of (5.3), giving as before the result in the form $\sqrt{Z^2 + 4a^2} = A + iB$

Substituting the values of X and Y we find $P = -40$, $Q = 58$, $A = 3$, and $B = 7$. Hence we have

$$z = \frac{1}{2}(1 + 21i) + \frac{1}{2}(3 + 7i) = 2 + 14i$$

Hence on the z plane, we have the point $x = 2$, and $y = 14$.

We then substitute in the formula for u and v on the z plane (see V 2) and find

$$\left.\begin{aligned} u &= -14.8 \\ v &= 1.4 \\ V &= \sqrt{u^2 + v^2} = 14.866 \end{aligned}\right\} \; z \text{ plane}$$

whence

Next we have from the equation of transformation,

$$\frac{dZ}{dz} = 1 + \frac{a^2}{z^2}$$

$$z^2 = (2 + 14i)^2 = 56i - 192$$

$$1 + \frac{a^2}{z^2} = 1 + \frac{100}{z^2} = 1 - .14i - .48 \quad \text{[see Division A VI (9.2)]}$$

whence

$$\frac{dZ}{dz} = .52 - .14i$$

This is a vector of which the scalar length is

$$r = \sqrt{\overline{.52}^2 + \overline{.14}^2} = .5385$$

This is then the inverse numerical ratio between the V on the z plane and that on the Z plane. Whence we have

$$V_Z = 14.866 \div .5385 = 27.61$$

We must now find the inclination of V_Z to X in order to find the X and Y components. To this end we find α the vector angle of V_Z from $\tan\alpha = v/u = -1.4/14.8$, whence

$$\alpha = 174^0\ 36'$$

Next we have θ, the vector angle of dZ/dz from $\tan\theta = -14/52$ or

$$\theta = -15^0\ 04'$$

whence $\qquad (\alpha + \theta) = 159^0\ 32'$

and
$$\left.\begin{aligned} u &= 27.61\cos(\alpha+\theta) = -25.862 \\ v &= 27.61\sin(\alpha+\theta) = 9.655 \end{aligned}\right\} Z \text{ plane}$$

These values are readily checked from (5.11). We have as above, $P = -40$, $Q = 58$, $A = 3$. Substituting these values with X, Y, and U as before, we find, numerical values,

$$\left.\begin{aligned} u &= -25.862 \\ v &= 9.655 \end{aligned}\right\} Z \text{ plane}$$

The same values as derived by transformation from the z field.

6. Flow of Indefinite Field about any Inclined Line. Consider the transformation

$$Z = z + \frac{a^2 e^{2i\alpha}}{z} \tag{6.1}$$

applied to a circle of radius a. In Fig. 43 the vector z is represented by OP, and the vector a^2/z by OQ. Then the vector $a^2 e^{2i\alpha}/z$ will be represented by the vector OQ rotated through the angle $+2\alpha$. Let LOM be a line inclined at an angle α to X. Then the rotation of OQ through the angle $+2\alpha$ will carry it to a position OQ_1 such that $POM = Q_1OM = (\alpha - \theta)$. But the addition of the two vectors OP and OQ_1 will give a point on the line LM. Hence the combination of the two vectors as in (6.1), z representing any point P on the circle, will result in transforming P to the line LM, and it is readily seen as in 3 that for the entire circle this will give the line AB of length $4a$.

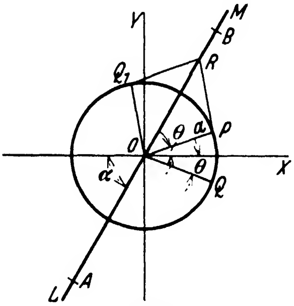

Fig. 43.

This equation will therefore transform the circle of radius a into a straight line of length $4a$ inclined at an angle α with the axis of X. The transformations of 3 and 5 are thus only special cases of this more general form.

If then we apply this transformation to the circle and to the field of flow parallel to X about the circle, as in 5, we should obtain the field of flow parallel to X about a straight line inclined at an angle α to the direction of flow.

To this end we take the equation: $w = -U\left(z + \frac{a^2}{z}\right)$ (6.2)
for the field of flow and expand this together with (6.1) by putting

$$z = (x + iy)$$

and

$$e^{2i\alpha} = \cos 2\alpha + i \sin 2\alpha.$$

This will give:

$$\varphi = -Ux\left(1 + \frac{a^2}{r^2}\right) \qquad (6.3)$$

$$\psi = -Uy\left(1 - \frac{a^2}{r^2}\right) \qquad (6.4)$$

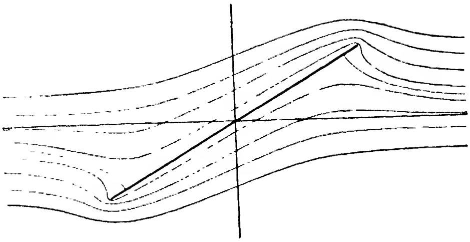

Fig. 44.

$$X = x\left(1 + \frac{a^2}{r^2}\cos 2\alpha\right) + \frac{a^2 y}{r^2}\sin 2\alpha \qquad (6.5)$$

$$Y = y\left(1 - \frac{a^2}{r^2}\cos 2\alpha\right) + \frac{a^2 x}{r^2}\sin 2\alpha \qquad (6.6)$$

In (6.5) and (6.6) if we put $\alpha = 0$ the values reduce to those of 4 while if we put $\alpha = 90^0$, they reduce to those of 5, showing again that these are merely special cases under (6.1) as the general formula of transformation.

The field for $\alpha = 30^0$ is shown in Fig. 44.

CHAPTER VII

PARTICLE PATHS, FIELDS OF FLOW RELATIVE TO AXES FIXED IN FLUID

1. Relative Motions in a Field of Flow—Stream-Lines and Particle Paths. In a field involving the relative motion of a solid and a fluid, we may refer the motion either to a set of axes fixed in the body or to a set assumed as fixed relative to the fluid at a great distance from the body. Thus standing on a bridge and watching the water flow by about one of the bridge piers, we associate ourselves with the solid obstacle—the pier—and refer the motion to axes considered as fixed in that body. Again if we are standing on a bridge over an arm of a lake in which the water is at rest relative to the earth and note the motion set up in the water as a small boat approaches, passes the point of observation and recedes in the distance, we associate ourselves with the earth or with the outlying body of still water and refer the motion to axes considered as fixed relative to the outlying and undisturbed fluid.

Again, we may be concerned with two different kinds of path or line, as follows:

(a) A line such that at any one instant of time, its direction at every point is in the line of fluid motion at that point. The name "*stream-line*" has already been given to a line of this character.

(b) A line which traces the path of an individual particle of the fluid through successive instants of time.

Line (a) gives, therefore, an instantaneous picture, as it were, of the direction of motion of a series of particles making up, as a whole, *an apparently continuous line of flow.*

Line (b), on the other hand, gives, through successive instants of time, *the path of a single particle of the fluid.*

If we imagine it possible to take an instantaneous photograph of the individual particles of a field of fluid motion, then by joining up, end to end, a series of minute traces of the motion of such particles, the photograph would give a line of the character of (a). On the other hand, if we assume that we can mark or identify an individual particle of the fluid, then a series of successive exposures on this one particle will give elements of a path of the character of (b).

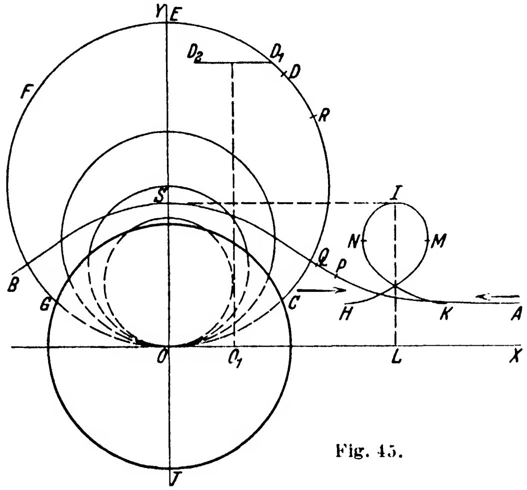

Fig. 45.

These paths may or may not be the same, as we shall proceed to show. Thus in Fig. 45 let CTG be a circle about which a two-dimensional sheet of fluid is flowing, say from right to left. We thus assume the obstacle fixed and the fluid flowing toward and around it. If we then trace the path of an individual particle, we shall have a path such as AB. This shows the path of the particular particle as it approaches, flows around and then recedes from the obstacle in the down stream direction.

Likewise if we could take an instantaneous photograph of the entire collection of particles making up the moving field we should find that one series of traces of such particles, beginning at A for example, would join, as it were, end to end and make up exactly the same line. The reason for this is readily seen. Thus, having given the circle and the remote field velocity U, the particular motion of a particle at Q will be wholly determined by the location of Q relative to the center of the circle O. Or in other words, the motion at Q is determined wholly by the field conditions at that point. Thus when the particle P, just behind Q, reaches the location Q, it will be under the same field control as was Q at that point, and in consequence will move as did Q at that point.

Hence, at every point in the field surrounding the circle the motion is single valued and determinate and as any particular particle reaches any particular point, it must move in the direction and with the velocity belonging to that point. It thus follows that a particle at A must follow along and ultimately pass through points P and Q and so along the entire path $A B$.

It thus appears that, in the present case, the stream-line and the particle path are the same.

Suppose on the other hand that we assume axes fixed relative to the distant fluid and note the movements in the fluid as the circle approaches and moves by, say from left to right. In this case the field conditions determinate of the movement of a particle of the fluid, move with the circle and hence are not constant at any one point. As will be shown later, the stream-lines—the result of an instantaneous photograph of the circle and of the particles composing the fluid field about it—will be composed of circles, as in Fig. 45, tangent to X and with their centers on Y.

On the other hand the path of an individual particle will be something like $H I K$, with the particle at I when the circle is moving past with center at L.

The stream-line and the particle path are, therefore, in this case, entirely different. The reason for this again, readily develops as follows:

Let u and v be the velocities along X and Y with axes fixed relative to the circle, as in V 2. Let u' and v' be the corresponding velocities in the present case. Then we shall have

$$u' = u + U$$
$$v' = v$$

The velocities u and v are as given in V 2 and hence at any point in the field at a given location relative to the circle, we shall have definite and known values of u' and v'. The path of an individual particle, starting at H, for instance, will then be the result of the joint action of these two component velocities u' and v'.

With the particle at H and the circle far away on the left, the particle is without movement. As the circle approaches bringing its "field" with it, there will begin to develop an excess of pressure on the left and below H resulting in values of u' and v' to the right and upward, and starting off the path as shown. It may be noted, however, that at the start with the body at a great distance, v' will be infinitesimal compared with u' and in consequence the path will start off tangent to the axis of X, gradually curving upward as the body approaches. With continued approach, v' increases relative to u' and finally at M, u' has become zero and beyond changes sign, the motion continuing upward, thus determining the path $M I$. At I, v' has become zero and

u' has gained its maximum value directed to the left. This will be at the instant where the center of the circle is at L. Following this the conditions succeed in reverse order and the path INK is traced. These various values of u' and v' are readily verified by reference to the formulae for u and v in V 2. It is obvious, also, that if H and A represent particles originally at the same distance from X, then LI must equal OS. In fact, the entire path HIK may be considered as the path ASB transformed by the application of a velocity $+U$ along the direction of X. On the other hand, consider, at any instant, a particle at R. This will have the u' and v' velocities appropriate to that location relative to O, the former to the left and the latter upward, thus determining a movement in the direction of the curve CRD at R. Similarly at D there will be a different particle at a different location relative to O and subject to the u' and v' appropriate to that point, and thus determining motion along the curve in the direction of the tangent at D. Thus every particle on the line $CDEFG$, at the instant of time when the circle is at this particular location in the field, will be moving in the direction of the tangent to the curve at the location of that particle. It will thus fill the definition of a stream-line. It is made up of a series of particles, all at that particular instant moving in the direction of the line.

It is easily shown, however, that this curve cannot be a particle path. Thus at the instant in question the particle at D is moving in the direction of the tangent at D and this will, in an element of time dt, carry it to a point D_1 near to D and near to the second order, to the curve DE. By this time, however, the circle with its field will have moved to the right and the field conditions at D_1 will no longer be those which prevailed when the particle was at D, and, therefore, they are no longer those which will continue to determine movement along the curve DE. When the particle at D has reached the near by position D_1, O will be at some point O_1 and the field conditions at D_1 will have become those which *now* belong to some point D_2 and these will determine a motion entirely different from that required to hold the particle at D_1 on the curve DE. The stream-line $CDEFG$, in such case, is not, therefore, at the same time a particle path.

Turning now to the particle path HIK, the direction at points H, M, I, etc. belong to different positions of the circle along the axis of X and hence to different instants of time. Conversely these directions along the particle path belong to no one location of the circle and to no one instant of time and hence cannot meet the conditions for a stream-line as defined.

It appears, therefore, that if the axes of reference are fixed relative to the body and the fluid "field" which it produces, the stream-lines and the particle paths are the same. If, on the other hand, the axes of reference are fixed in the fluid through which there passes a body

with its moving "field" of control, the stream-lines and the particle paths will not be the same.

Again a field of flow in which the conditions at a point fixed relative to a solid obstacle do not change with the time, has been defined as a field of steady flow or steady motion; while a field in which the conditions do change with the time is a field of unsteady flow. From the preceding it will be seen that in a field of steady flow, the stream-line and the particle path will be the same, while in a field of unsteady flow, they will not be the same.

2. Path of a Particle Relative to Axes Fixed in the Fluid. In certain special cases it is possible to integrate the equations of motion relative to axes fixed in the fluid and thus to derive an equation to the path[1]. The mathematical procedure is somewhat complex and the space required for adequate presentation can hardly be justified in the present work. It may be of interest, however, to develop a method whereby, having given a field of flow, it becomes a simple matter to work out, by a combination of graphical and numerical methods the path for any particle in the field.

Imagine two fields of flow, one lying above the other and with axes and origin coinciding in the projection on an $X\,Y$ plane. Let the first represent a flow in an unobstructed field and in the direction of $-X$ with the linear field velocity U. Let the second represent a flow the same as for the first at a great distance, but with an obstruction of some character at the origin. Next suppose two particles, one in each field, identical in projection on $X\,Y$ and at a great distance from the origin, to start at the same time. Then the path we desire will be that of the second particle relative to the first. Each particle will move with the velocities u and v appropriate to its field; the first with $u = -U$ and $v = 0$ and the second with u and v as determined by the conditions in the field. Then denoting the relative velocities by u' and v', we have:

$$u' = -U + u$$
$$v' = v$$

The path desired will be then, that of a point moving with these coordinate velocities u' and v'. Hence considering x' and y' as the coordinates of a point on such a path, we have

$$\frac{d\,x'}{d\,t} = -U + u$$

$$\frac{d\,y'}{d\,t} = v$$

whence

$$\left.\begin{aligned} x' &= -Ut + \int u\,dt \\ y' &= \int v\,dt \end{aligned}\right\} \qquad (2.1)$$

---

[1] Morton, W. B., *Proc. Royal Society London*, 89 A, p. 106. Maxwell, J. Clerk, "Coll. Papers", vol. II, p. 208. Havelock, *Univ. Durham Phil. Soc. Proc.*, vol. IV, 1911. Rankine, W. J. M., *Royal Society London Phil. Trans.*, 1864.

In Fig. 46 let ML be any stream-line mapped out in the field of flow. A particle starting from M, at a great distance on the right, will, in a certain time arrive at P_2. The same particle, starting at the same point in the same field without obstruction, would, in the same time arrive at some point P_1. Then

$$x' = P_1 B \quad \text{and} \quad y' = B P_2$$

Knowing the location of the particle at M, and knowing $A P_2$, we have immediately $\quad y' = BP_2 = A P_2 - A B$

Likewise $\quad x' = P_1 B = P_1 M - BM$

But with the points P_2 and M known, the distance BM is known.

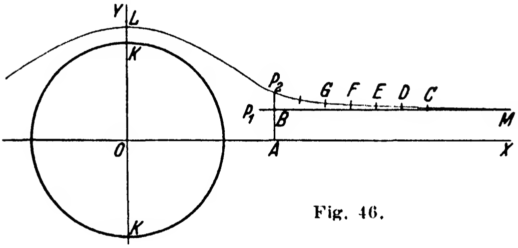

Fig. 46.

Now suppose that M is so far from KK that the influence of the obstruction on the movement of particle two up to that point has been negligible. Then the two particles may be assumed to be sensibly together at M. Let t be the time required for particlé one to move from M to P_1 or for particle two to move from M to P_2. Then $\quad P_1 M = Ut$

To find $P_1 M$ we must then find t, the time required for particle two to go from M to P_2.

Let C, D, E, F, be successive points on $M P_2$. We assume full information with regard to the field of flow as in the diagram, and including a knowledge of the functions φ and ψ. We can thus find the value of φ at these various points on $M P_2$, or otherwise, we may locate such points at any desired series of successive values of φ. But φ has the character of a line integral and we may therefore write

$$\Delta \varphi = \Delta s \frac{\Delta s}{\Delta t} \tag{2.2}$$

That is, Δs is an element of the path, such as CD, and $\Delta s/\Delta t$ is the mean velocity over such path. Then the product will be $\Delta \varphi$ as in (2.2). This gives:

$$\Delta t = \frac{(\Delta s)^2}{\Delta \varphi} \tag{2.3}$$

If then these points are taken sufficiently near so that the elements of the path CD, DE, etc. may be taken sensibly as straight lines and the length Δs thus measured or computed, we shall have, through (2.3), means for computing Δt, the time required for the moving particle to traverse these various elements of the path $M P_2$. By summation, then, we may determine the time for any part of the path such as $C P_2$.

We have now to consider the time required for the path MC, where C is a point far enough from the origin to justify the assumption that the path MC differs insensibly from what it would be if the obstruction KK were a circle of the same general order of magnitude.

For the case of a circle, we have in general, [see V (2.4) (2.6)]

$$u = -U\left(1 - \frac{a^2(x^2 - y^2)}{r^4}\right)$$

Where x is large relative to a and y, we may take r as sensibly equal to x and in such case the above value of u (numerical values) reduces to

$$u = U\left(1 - \frac{a^2}{x^2}\right) = \frac{dx}{dt}$$

whence
$$U\frac{dt}{dx} = \frac{x^2}{x^2 - a^2}$$

and
$$Ut = \int_{x_2}^{x_1} \frac{x^2\,dx}{x^2 - a^2}$$

This expression will then give the value of Ut for any stretch of the path far from the origin. The above expression under the integral may be resolved into partial fractions as follows:

$$a\left(\frac{x^2}{a(x^2 - a^2)}\right)dx = a\left[\frac{dx}{a} + \frac{dx}{2(x-a)} - \frac{dx}{2(x+a)}\right]$$

These may be separately integrated and give

$$t = \left[\frac{x}{U} + \frac{a}{U}\cdot log\sqrt{\frac{x-a}{x+a}}\right]_{x_2}^{x_1}$$

or putting in limits

$$t = \frac{x_1 - x_2}{U} + \frac{a}{U}\,log\cdot\sqrt{\frac{(x_1 - a)(x_2 + a)}{(x_1 + a)(x_2 - a)}} \tag{2.4}$$

From the form of (2.4), it is seen that the term $(x_1 - x_2)/U$ is the time which would be required for a particle moving in a free field under the velocity U while t is the time actually required in the obstructed field. The second term represents, therefore, the retardation due to the obstruction and this multiplied by U will give the advance of one particle relative to the other for the distance $(x_1 - x_2)$.

If x_1 is put at ∞, we shall then have

$$\text{advance} = a\,log\sqrt{\frac{x_2 + a}{x_2 - a}} \tag{2.5}$$

By way of illustration, with $a = 10$ and x_2 as in the following table, the advance of the particle in the free field over that moving in the field with the circle as an obstruction, will have values as follows:

x_2	*Advance*	x_2	*Advance*
100	1.000	40	2.553
80	1.256	20	5.494
60	1.682		

With $a = 10$, the values for the relative advance at $x_2 = 20$ and 40 are, of course, less accurate than for the larger values. It is interesting to note that at a distance of 10 times the radius, the relative advance is one-tenth the radius. That is, in coming from ∞ to a distance of ten radii from the circle, a particle moving near the axis of X will be retarded a distance equal to one-tenth the radius, relative to the same particle moving in a free field.

Now referring back to Fig. 46, it is seen that (2.5) will give the relative displacement at a point C appropriately chosen. Then from this as a starting point, the remainder of the process is seen to comprise the following steps.

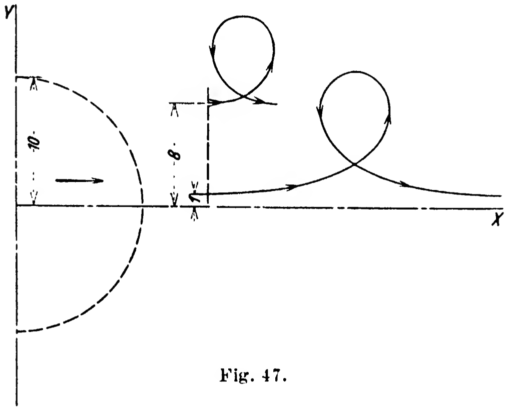

Fig. 47.

(1) The division of the path into a series of elements CD, DE, etc., taken at equal intervals for φ, or selected otherwise, and the values of φ and of $\triangle\varphi$ found.

(2) The measurement of the intervals $\triangle s$ and thence the values of $\triangle t$ as in (2.3).

(3) Then for each element of the path the value of $(U \triangle t - \triangle x)$ will give the relative displacement or advance for such element.

(4) The successive summations of these elements of the advance, corrected for the advance at the point C, will then give the total advance similar to $B P_1$ for the point P_2.

(5) The treatment for that part of the path beyond the obstruction is, of course, entirely similar and need not be considered in detail.

(6) These various values of $B P_1 = x'$ combined with $B P_2 = y'$ will then give the entire path for particle two relative to particle one.

In Fig. 47 are shown two paths for the case of a circle of radius ten, one starting at a distance of one unit from X and the other at a distance of eight.

3. Derivation of Velocity Potential and Stream Function for Fields of Motion Relative to Axes Fixed in the Fluid. We assume that we have given the functions φ and ψ for motion relative to axes fixed in the body about which the fluid is moving. Let φ' and ψ' denote the corresponding functions for motion relative to axes fixed in the fluid. It is understood that by axes fixed in the fluid we mean fixed relative to the fluid field at a great distance and entirely beyond any sensible influence due to the body about which the fluid is moving. In the case

of motion relative to axes fixed in the body, we assumed the general fluid field as moving with a velocity U from right to left or in the direction of $-X$ and hence designated by $-U$. In the present case with the axes fixed in the fluid, we shall assume the body moving from left to right, with the velocity U, and therefore designated by U.

Let u' and v' denote the velocities in the present case corresponding to u and v with the functions φ and ψ. Then evidently v and v' are the same while the difference between u and u' is obviously the velocity U.

Hence,
$$u' = \frac{\partial \varphi'}{\partial x} = \frac{\partial \psi'}{\partial y} = u + U \tag{3.1}$$
$$v' = \frac{\partial \varphi'}{\partial y} = -\frac{\partial \psi'}{\partial x} = v \tag{3.2}$$

Then from (3.1) $\varphi' = \int (u + U)\, d\,x = \int \frac{\partial \varphi}{\partial x} d\,x + \int U\, d\,x$

or
$$\varphi' = \varphi + U x \tag{3.3}$$
$$\psi' = \int (u + U)\, d\,y = \int \frac{\partial \psi}{\partial y} d\,y + \int U\, d\,y$$

or
$$\psi' = \psi + U y \tag{3.4}$$

These are partial integrations, that is, integrations for φ' relative to x and for ψ' relative to y. Hence to make sure of the complete form of φ' and ψ' we should also integrate φ' with reference to y and ψ' with reference to x, as in (3.2). If this is done we shall find simply $\varphi' = \varphi$ and $\psi' = \psi$ which shows that the integrations in (3.1) give the complete form while those in (3.2) give only a partial form.

Hence (3.3) and (3.4) give the complete values of φ' and ψ' in terms of φ and ψ.

It is clear that φ' and ψ' have the same mathematical characteristics as φ and ψ and that their derivatives have the same relations as in Division A VII 5. Hence if we denote the potential function for φ' and ψ' by w', we shall have, as for φ and ψ,
$$w' = \varphi' + i\psi' \tag{3.5}$$

Substituting in (3.5) for φ' and ψ' as above, we have
$$w' = (\varphi + i\,\psi) + U\,(x + i\,y)$$

or
$$w' = w + U z \tag{3.6}$$

4. Field of Flow for a Thin Circular Disk Moving in Its Own Plane in an Indefinite Fluid Field. This is the converse of V 2 and we may derive the functions φ', ψ' and w' as in 1, directly from the results of that section. Taking the values of φ and ψ from V (2.7) (2.8) (2.9) we have
$$\left.\begin{aligned} \varphi' &= -U\frac{a^2 x}{r^2} = -U\frac{a^2 \cos\theta}{r} \\ \psi' &= U\frac{a^2 y}{r^2} = U\frac{a^2 \sin\theta}{r} \end{aligned}\right\} \tag{4.1}$$
$$w' = -U\frac{a^2}{z}$$

The stream-lines will be given by

$$\psi' = \text{const.}$$

or $$U a^2 y = (x^2 + y^2)\, C, \text{ where } C = \text{constant value of } \psi'.$$

This reduces to the form

$$x^2 + \left(y - \frac{U a^2}{2\,C}\right) = \left(\frac{U a^2}{2\,C}\right)^2$$

These are circles with radius $U a^2/2\,C$ and all tangent to X at the origin or center of the circle. Obviously that part of the line inside the circumference of the circle will have no physical existence.

This field is shown in Fig. 45.

If these values of φ' and ψ' are compared with the values of φ and ψ for a doublet as in IV 10, it will be seen that if we put $M = U a^2$ or $a = \sqrt{M/U}$, they become the same. This means that the field of the circle of radius a, moving in its own plane with a velocity U is the same as that of a doublet of moment $M = U a^2$. Naturally, for the circle, only that part of the field outside the circumference has a physical existence.

A comparison of the values of φ' in 3 and of φ in V 2 shows that, in effect, the procedure of 3 amounts to a cancellation of the field of rectilinear flow along $-X$. Hence if we start with the doublet field of IV 10, add a rectilinear flow giving the field of V 2 and then cancel the rectilinear field, we shall naturally return to the doublet field, as at the start.

It thus appears that since a doublet placed in an indefinite stream will impose on such stream the field for flow about a circle, then a circle moving in the stream at rest will reproduce the initial field of the doublet.

For the energy of the field, we have from Division A IX (2.3)

$$E = -\frac{\varrho}{2}\int \varphi\, d\,\psi$$

Here, in integrating around the circle, $r = \mathit{constant} = a$ and this gives.

$$E = \frac{\varrho}{2}\, U^2 a^2 \int_0^{2\pi} \cos^2\theta\, d\,\theta = \frac{\varrho}{2}\,\pi\, a^2\, U^2$$

where E is the energy per unit thickness of cylindrical disc, or in general, per unit length of a circular cylinder moving $\perp$ to its length.

But $\varrho\,\pi\, a^2$ is the mass of a volume of fluid equal to the volume of unit length of the cylinder. Hence the energy in the field is the same as that of a mass of fluid equal to that of the disc, and moving with the same velocity U. The virtual or added mass due to the motion is therefore, equal to that of a volume of fluid equal to the volume of the disc.

5. Field of Flow Produced by a Circle Combined with Vortex Flow Moving in an Indefinite Fluid Field. This is the inverse of V 4. There the axes were fixed in the body, here they are fixed in the fluid. The general formulae of 3 apply and we thus derive as follows:

$$\varphi' = -\frac{Ua^2 x}{r^2} + k\,\theta = -\frac{Ua^2 \cos\theta}{r} + k\,\theta$$

$$\psi' = \frac{Ua^2 y}{r^2} - k \log\frac{r}{a} = \frac{Ua^2 \sin\theta}{r} - k \log\frac{r}{a}$$

$$w' = -U\,\frac{a^2}{z} - i\,k \log z$$

The character of the field is shown in Fig. 48. In this and all other cases of this character it should be remembered that the diagrams of Figs. 31 and 48 refer to the same physical problem, but in the one case we, as observers, have identified ourselves with the body (Fig. 31) and in the other with the fluid medium (Fig. 48).

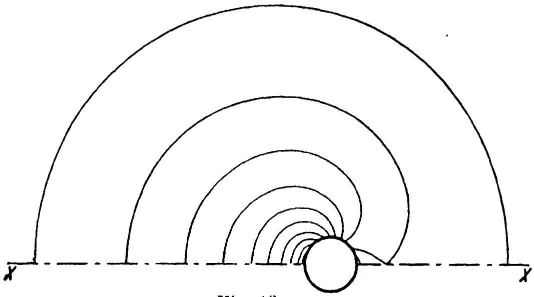

Fig. 48.
(Movement $\perp$ to XX.)

6. Field of Flow for a Straight Line Moving at Right Angles to Itself in an Indefinite Fluid Field. This is the converse of VI 5 and we derive the various functions in the same general manner as in the preceding paragraph. We thus have

$$w' = U\left[Z - \sqrt{Z^2 + 4a^2}\right] \qquad (6.1)$$

$$\varphi' = U\left[X - \left(\frac{P + \sqrt{4X^2Y^2 + P^2}}{2}\right)^{1/2}\right] \qquad (6.2)$$

$$\psi' = U\left[Y - \left(\frac{-P + \sqrt{4X^2Y^2 + P^2}}{2}\right)^{1/2}\right] \qquad (6.3)$$

$$u' = U\left[1 - \frac{X(P + Q + 2Y^2)}{2AQ}\right] \qquad (6.4)$$

$$v' = UY\left(\frac{P + Q - 2X^2}{2AQ}\right) \qquad (6.5)$$

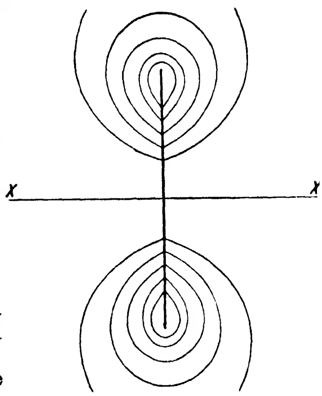

Fig. 49.

Equations (6.4) and (6.5) will give a velocity $u' = U$ for all points on the line, with a Y velocity the same as in VI 5. The same infinite value of the velocity appears at the ends of the line, implying a failure of the equations at that point to represent a possible set of physical conditions. The character of the stream-lines in this field is shown in Fig. 49.

CHAPTER VIII

DERIVATION OF POTENTIALS BY INDIRECT METHODS

1. The Derivation of Velocity Potential and Stream Functions by Indirect Methods. The fields of flow which have been studied thus far have been developed from the obvious elementary fields for rectilinear flow in an unobstructed field, for a source (or sink) and for a vortex flow, through combination and conformal transformation. We may now consider a somewhat indirect method which may sometimes yield useful results.

We know that to every developed form of the function $w = f(z) = f(x + iy)$ (see Division A VII 5) there exists a pair of functions φ and ψ and that indifferently one of these may be taken for a velocity potential and the other will serve as a stream function. We may thus say that to every developed form of $f(x + iy)$ there are two (mathematically) possible fields of flow; and *vice versa* to every field of flow there will correspond a function $w = f(x + iy)$ with its pair of functions φ and ψ.

We may now ask what are the physical conditions of a field of flow which must be met by the functions φ and ψ. These may be grouped as follows.

(1) The fluid is assumed incompressible and the motion irrotational.

(2) If the axes are fixed in the body, the velocity everywhere at infinity must be the velocity of translation U or $-U$, according to the convention of signs. Likewise, the boundary or contour of the body must be a stream-line, or otherwise the component velocity along the normal at all points around the boundary must be zero. Instead of going to infinity, it may be possible to assume at a given part of the field, a given velocity or distribution of velocities, without inquiring as to how this particular condition came about. So long then as such velocity or distribution of velocities meet the requirements of the velocity potential, the remainder of the field will show properly the conditions of flow.

(3) If the axes are fixed in the fluid at infinity, the velocity everywhere at infinity must be zero. Likewise, the component velocity of a fluid particle at any point on the boundary of the body, taken along the normal to the contour, must equal the component velocity of translation taken along the direction of the same normal.

The same special assumption regarding velocity in a part of the field not at infinity may be made here as in the preceding case.

Conditions (1) are, of course, the general conditions which we have assumed for fluid motion and are fulfilled by the functions φ and ψ resulting from any development $f(x + iy)$. Conditions (2) and (3) are the so-called boundary conditions and relate to the special case in hand.

We may say, therefore, that the solution for any special case is to be found in a pair functions φ and ψ which will fulfill the boundary conditions imposed by the nature of the case. There is, unfortunately no general method of deriving the functions having given the boundary

conditions and in consequence it is not possible in general, to proceed in this manner to a final solution. It is possible, however, to proceed in the inverse direction by assuming various forms of the function $w = f\,(x + iy)$ and thus develop pairs of functions φ and ψ and then examine the boundary conditions which they will fulfill. In this way useful results may be developed.

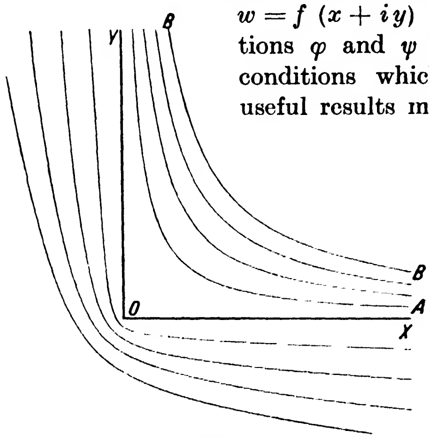

Fig. 50.

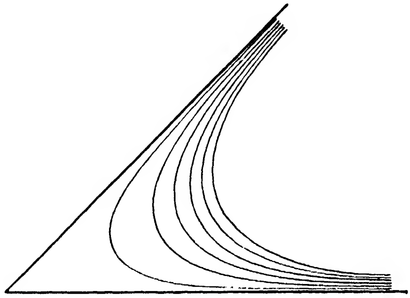

Fig. 51.

For example if we take $w = A z^2$. We shall then have, as in Division A VII 7

$$\varphi = A\,(x^2 - y^2)$$
$$\psi = 2A\,xy$$

If we take ψ for the stream-lines, we shall have a field as in Fig. 50 (taking the part only in the positive quadrant). It is seen here that u becomes ∞ with $x = \infty$ and $v = \infty$ with $y = \infty$. It is of interest to note that this is due to the fact that the spacing between the stream-lines at ∞ is 0 and hence the velocity must be ∞ in order to provide for a finite flow between two adjacent lines. In such a case, however, in a part of the field $A\,B$ far from O, the stream-lines will be practically parallel to X and we may assume a uniform velocity of flow across a section $A\,B$. The hyperbolae will then give the stream-lines for a flow guided by the hyperbola BB and the two axes X and Y meeting in a right angle at O.

More generally assume $w = A z^n$. Then passing to polar coordinates we have $w = A r^n\,(\cos\theta + i \sin\theta)^n = A r^n \cos n\theta + i\,A r^n \sin n\theta$

whence $\varphi = A r^n \cos n\theta$

$\psi = A r^n \sin n\theta$

The preceding example is a special case of these general values where $n = 2$.

It will be readily seen that the curves given by

$$r^n \sin n\theta = \text{const.}$$

are hyperbolic lines of flow for a fluid flowing around the angle between two lines (or walls) inclined at an angle π/n. Thus in Fig. 51 is shown

the field of flow for an angle of 45°, and in Fig. 50, that for an angle of 270° or around the outside of a square corner.

As a further illustration of the Inverse Method of Approach, take, as in V 2 the case of an infinite fluid sheet flowing past a circular boundary. The axis of X and the circular boundary must form part of the stream-line system. Taking these as the line for $\psi = 0$, it follows that the

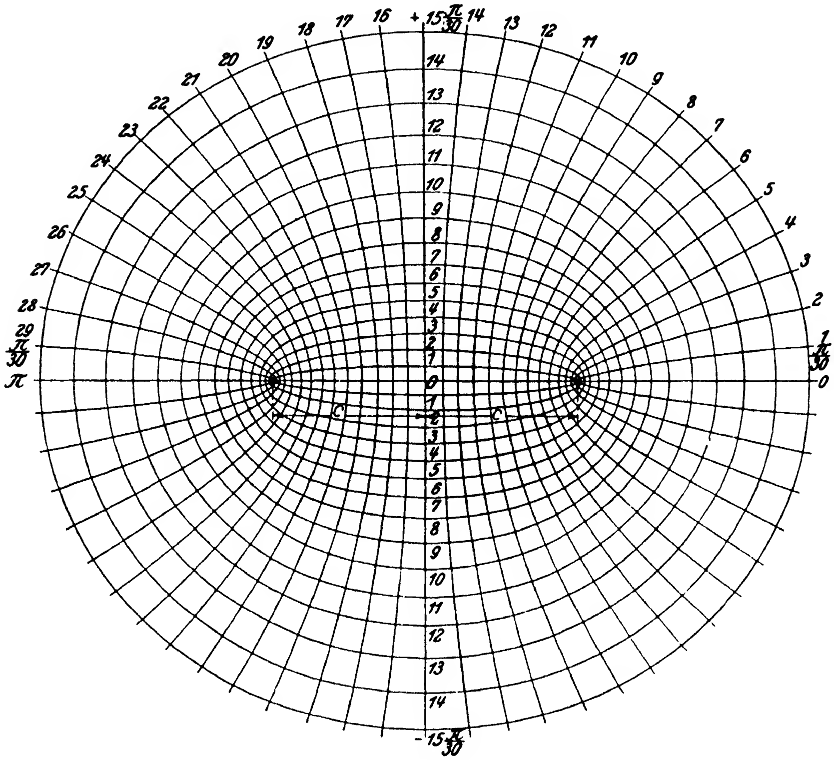

Fig. 52.
(From PRÁŠIL, Technische Hydrodynamik, 2. Aufl., 1926.)

expression for ψ will contain y and $(r^2 - a^2)$ as factors. These multiplied by a constant and by some at present unknown function of xy will then give the form of the function ψ. We may therefore write

$$\psi = A \frac{y\,(r^2 - a^2)}{f\,(x, y)}$$

where $r^2 = (x^2 + y^2)$ and for convenience we put the unknown function of x, y in the denominator.

A value $\psi = 0$ will then give, as the stream-line, $y = 0$ and $r^2 = a^2$ or the axis of X and the circle of radius a. It is then necessary to determine $f\,(x, y)$ in such way as to give the velocities at ∞, $u = - U$ and $v = 0$.

If then we can guess or hit upon $f(x, y) = x^2 + y^2 = r^2$, we shall have

$$\psi = A y \left(1 - \frac{a^2}{r^2}\right)$$

whence we may derive φ as in Division A VII 7 and thus the entire field as in V 2.

2. Field of Flow through an Opening in an Infinite Rectilinear Barrier. This case will again illustrate the indirect method of approach. In Division A I 3 as an example of the inverse relation $z = F(w)$, the function $z = c \cos w = c \cos(\varphi + i\psi)$ was employed. As there shown, this function gives the two equations

$$\frac{x^2}{c^2 \cosh^2 \psi} + \frac{y^2}{c^2 \sinh^2 \psi} = 1 \tag{2.1}$$

$$\frac{x^2}{c^2 \cos^2 \varphi} - \frac{y^2}{c^2 \sin^2 \varphi} = 1 \tag{2.2}$$

With $\psi = const.$ for a series of values, we have a series of ellipses and with φ similarly, we have a series of hyperbolas, giving a field as in Fig. 52.

If the ellipses are taken for the stream-lines, we have the case of a fluid circulating about an elliptic boundary. In the extreme case this becomes a straight line of length $2c$.

If the hyperbolas are taken for the stream-lines, we have the case of a fluid flowing through an opening of length $2c$ between two indefinite rigid lines.

Since the space between the stream-lines continuously widens out with departure from the origin, the velocity will continuously decrease and become zero at infinity. In the same way with the ellipses as stream-lines, the space between the lines for equal increments of ψ continually increases with removal from the origin, and the velocity of circulation will clearly decrease to zero at infinity.

3. Field Produced in an Indefinite Fluid Sheet by the Movement of a Thin Lamina in its Own Plane. This case is similar to that of VII 4 but more general in character. It will furnish a further illustration of the inverse method. Let ABC, Fig. 53, denote the boundary. Then at a point P, the velocity of a particle along the normal will be $U \cos \theta = U\, dy/ds$. But this velocity will also be measured by $\partial\psi/\partial s$ (see Division A VIII 4). Hence we may write $\frac{\partial \psi}{\partial s} = U \frac{dy}{ds}$

Integrating we have $$\psi = U y + \text{const.} \tag{3.1}$$

This is, of course, only a particular value of ψ, applying along the boundary of the body. One reason why this cannot be a general value of ψ is seen in the fact that this is not a complete integration of $d\psi$, but only with reference to length along the contour s.

However, we may seek general solutions by assuming general forms for ψ and then by comparison with (3.1) find for what contour the general form will reduce to that of (3.1). In other words we may assume any form of ψ at will and then find the corresponding contour for which the conditions of (3.1) are fulfilled.

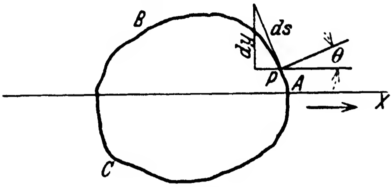

Fig. 53.

Thus suppose we should take for trial,

$$\psi = \frac{U a^2 \sin \theta}{r} \tag{3.2}$$

Then (3.2) is a general value which must agree with (3.1) on some contour. The condition for equality is

$$\frac{U a^2 \sin \theta}{r} = U y = U r \sin \theta. \quad \text{Whence } r = a$$

This is a circle of radius a. It thus appears that the form of function in (3.2) will, for a circular contour of radius a, meet the conditions of (3.1) and thus, providing the conditions at ∞ are met, it should serve to represent the field of flow for a circular lamina moving in an indefinite fluid sheet.

Comparison with VII 4 shows that this is the same form of function there developed and we need, therefore, pay no further attention to this particular case.

4. Field of Flow Produced by an Elliptic Contour Moving in the Direction of Its Axes. For the examination of this case it is a matter of convenience to make use of so-called elliptic coordinates. In Fig. 54 with a and b as radii, suppose two circles drawn from O as center. Draw any radius $O\,Q_1\,Q$ making an angle η with AB. Then from Q draw QE parallel to Y and through Q_1, FG parallel to X. These lines will intersect at a point P. Then from similar triangles, $PE/QE = FO/QE = Q_1O/QO = b/a$. This will determine P as a point on an ellipse with a and b as half diameters, as shown. We then have

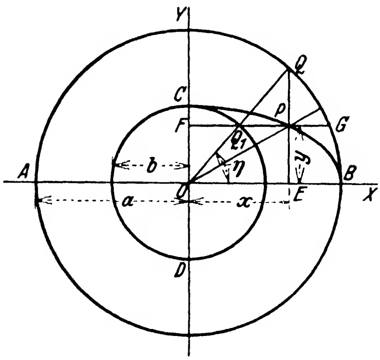

Fig. 54.

$$\left.\begin{aligned} x &= a \cos \eta \\ y &= b \sin \eta \end{aligned}\right\} \tag{4.1}$$

The angle η is thus common to both coordinates x and y. Next let:

$$\left.\begin{aligned} a &= c \cosh \xi \\ b &= c \sinh \xi \end{aligned}\right\} \tag{4.2}$$

Then referring to Division A I (10.2) and putting in the values for the hyperbolic sine and cosine, we find:

$$e^{\xi} = \frac{a+b}{c}; \qquad e^{-\xi} = \frac{a-b}{c} \tag{4.3}$$

$$\frac{a^2-b^2}{c^2} = 1$$

or
$$c = \sqrt{a^2-b^2} \tag{4.4}$$

whence from (4.2) and then (4.3)

$$\left.\begin{aligned} \xi &= tanh^{-1}\frac{b}{a} \\ \text{and} \qquad \xi &= log\sqrt{\frac{a+b}{a-b}} \end{aligned}\right\} \tag{4.5}$$

Now putting the above values of a and b into (4.1) we have

$$\left.\begin{aligned} x &= c\,cosh\,\xi\,cos\,\eta \\ y &= c\,sinh\,\xi\,sin\,\eta \end{aligned}\right\} \tag{4.6}$$

We have thus in (4.6) the values of the two coordinates x and y in terms of the single parameter c with value as in (4.4) and two new variables, ξ with value as in (4.5) and η an angle which may take values from 0 to $\pm\pi$.

From (4.6) we readily find

$$\frac{x^2}{c^2\,cosh^2\,\xi} + \frac{y^2}{c^2\,sinh^2\,\xi} = 1 \tag{4.7}$$

This is obviously the equation to an ellipse with semi-axes as in (4.2). The distance between the foci of this ellipse is $2\sqrt{a^2-b^2} = 2c$. It is thus independent of the variable ξ. It follows that with c constant and with a series of values for ξ, we shall have a series of confocal ellipses, for each one of which, ξ will be constant over the boundary. By the use of these new coordinates, ξ and η, we may therefore represent any point in the entire plane XY. The semi-axes of any such ellipse will be as in (4.2) where c is constant as determined from the basic ellipse and ξ is variable. Similarly the rectangular coordinates x and y will be as in (4.6). Thus with ξ and η given, the point is located by the values of x and y in (4.6). Inversely with x and y given and with c constant, we may, in (4.7), put $sinh^2\,\xi = cosh^2\,\xi - 1$ and then solve for $cosh\,\xi$ and thence find $sinh\,\xi$ and then η from (4.6) by dividing one equation by the other. The two coordinates ξ and η thus become known.

Now assume the function

$$\left.\begin{aligned} w &= \varphi + i\psi = Be^{-(\xi+i\eta)} \\ \text{or} \qquad \varphi + i\psi &= Be^{-\xi}\,(cos\,\eta - i\,sin\,\eta) \end{aligned}\right\} \tag{4.8}$$

whence
$$\left.\begin{aligned} \varphi &= Be^{-\xi}\,cos\,\eta \\ \psi &= -\,Be^{-\xi}\,sin\,\eta \end{aligned}\right\} \tag{4.9}$$

Equations (4.8) are independent of the form of the boundary and the coordinates ξ and η may, as we have seen, denote any point in the plane of XY. In (4.6), however, with a given value of c, y is the ordinate of the particular ellipse with semi-axes $c \cosh \xi$ and $c \sinh \xi$. That is, y in the equation $\psi = Uy + const.$ [see (3.1)], where the y belongs to (4.6), is restricted to the boundary of the ellipse (4.7). If then we write the equation

$$\psi = -Be^{-\xi} \sin \eta = Uy = Uc \sinh \xi \sin \eta \qquad (4.10)$$

We state the condition that the general value of ψ, as in (4.9), shall take the particular value for the boundary when ξ and η become the coordinates of a point on the ellipse of semi-axes a and b as in Fig. 54. This equation, however, contains the constant B, the value of which must be determined. To this end, we have from (4.2), (4.10)

$$-Be^{-\xi} = Ub$$

Also from (4.3) $\qquad e^{-\xi} = \dfrac{a-b}{c}$

whence $\qquad -B = \dfrac{Ubc}{a-b} = Ub\sqrt{\dfrac{a+b}{a-b}}$

Putting this in (4.9) we have finally

$$\left.\begin{aligned} \varphi &= -Ub\sqrt{\frac{a+b}{a-b}} \cdot e^{-\xi} \cos \eta \\ \psi &= Ub\sqrt{\frac{a+b}{a-b}}\, e^{-\xi} \sin \eta \end{aligned}\right\} \qquad (4.11)$$

In this equation we have the value of ψ for the field in general where ξ and η may designate any point, as above noted. For the ellipse, however, with a and b as semi-axes, the value of ξ in (4.5) gives $\psi = Ub \sin \eta$ or Uy as required by (4.10).

So far, therefore, as the conditions about the ellipse are concerned, (4.11) gives the correct value of the stream function ψ. We have now only to examine the conditions at ∞.

In (4.11) $\sin \eta$ can only vary between -1 and $+1$ and it is clear that for ξ large or ∞, ψ will become small or zero and further that the derivative of ψ in any direction will contain the term $e^{-\xi}$ and with ξ large or ∞, the velocity in any and all directions will become small or zero.

We conclude, therefore, that the value of ψ as in (4.11) correctly represents the field of flow for a lamina with an elliptic contour (semi-axes a and b) moving in the direction of its major axis in an indefinite fluid field.

If we suppose the motion in a direction parallel to the minor axis we may consider the ellipse turned so that a lies along Y and b along X. We still define η as the angle specified as before and measured from X. We must then put $\qquad x = b \cos \eta$

$$y = a \sin \eta$$

Then with the same assumptions and relations between a, b, c and ξ, we shall have, for the general coordinates x and y,

$$\left.\begin{aligned} x &= c \sinh \xi \cos \eta \\ y &= c \cosh \xi \sin \eta \end{aligned}\right\} \tag{4.12}$$

Again with the same assumption as in (4.8) we shall have the same result as in (4.9).

Then from the equation parallel to (4.10) we shall find

$$-Be^{-\xi} = Ua \text{ and thence}$$

$$\left.\begin{aligned} \varphi &= -Ua\sqrt{\frac{a+b}{a-b}}\, e^{-\xi} \cos \eta \\ \psi &= Ua\sqrt{\frac{a+b}{a-b}}\, e^{-\xi} \sin \eta \end{aligned}\right\} \tag{4.13}$$

The stream-lines given by fixed values of ψ in (4.11) or (4.13) are readily found by the following procedure.

(1) Assume a value of $e^{-\xi}$. This is equivalent to assuming the coordinate ξ.

(2) From the value of ψ find $\sin \eta$ and then $\cos \eta$.

(3) From the values of $e^{-\xi}$ and e^{ξ} find $\cosh \xi$ and $\sinh \xi$.

(4) Thence find x and y from (4.6) or (4.12).

These expressions for the field of flow produced by a lamina with an elliptical boundary are true no matter what the proportions of the ellipse and hence should hold when it becomes a circle at one extreme or straight line at the other. Thus for a straight line moving parallel to itself, $b = 0$ and ψ, as in (4.11), becomes zero and the field shows no motion whatever, as it should. On the other hand with a circle, $a = b$, $\xi = \infty$ and the value of ψ becomes indeterminate. Other methods for dealing with the circle as in VII 4 are therefore preferable.

Again for a straight line moving at right angles with itself, we have in (4.13) $b = 0$, whence,

$$\left.\begin{aligned} \psi &= Uae^{-\xi} \sin \eta \\ \varphi &= -Uae^{-\xi} \cos \eta \end{aligned}\right\} \tag{4.14}$$

This gives an alternate form for the results developed in VII 6. Computation shows that both forms give the same results. In this connection it must be remembered, however, that the a of VII 6, as the radius of the circle transformed into a straight line, is one-fourth the length of the line while the a of the present section is one half the length of the line.

For the energy of the field we have

$$E = -\frac{\varrho}{2}\int \varphi \, d\psi \qquad \text{[see Division A IX (2.3)]}$$

the integration being carried around the periphery. But on the periphery ξ is constant $= \xi_0$ and from (4.9)

$$\psi = -Be^{-\xi_0} \sin \eta = Ub \sin \eta$$

Also

$$\varphi = Be^{-\xi_0} \cos \eta = -Ub \cos \eta$$

$$d\psi = Ub \cos \eta \, d\eta$$

Then

$$E = \frac{\varrho}{2} U^2 b^2 \int_0^{2\pi} \cos^2 \eta \, d\eta = \frac{\pi \varrho U^2 b^2}{2}$$

or

$$E = \frac{\pi b^2 \varrho U^2}{2}$$

But $\pi b^2 \varrho$ is the mass of a circular disc of unit thickness, of density ϱ and with radius b ($\perp$ to the direction of the assumed motion) and E is therefore the energy of such a mass moving with the velocity U. It results that for a cylinder with elliptical section and of indefinite length, moving in a direction $\perp$ to its length and in the direction of the diameter $2a$ of its cross section, the energy per unit length will be equal to that of a circular disc of unit thickness and diameter $2b$ (the diameter $\perp$ to the direction of motion) moving with velocity U. This general result holds for any ellipse no matter what the excentricity. If we had taken the motion in the direction of b, the result would be given as above but with the exchange of a and b. Again if $a = b$ the ellipse becomes a circle with radius b. This reproduces the result of VII 4. In the other direction, if the ellipse shrinks to a line $2b$ moving $\perp$ to itself, we shall have the same result—for a flat band of indefinite length and of width $2b$ moving $\perp$ to its plane, the energy per unit of length is equal to that of a circular disc of diameter $2b$, of unit thickness, of density ϱ and moving with a velocity U.

CHAPTER IX
THREE-DIMENSIONAL FIELDS OF FLOW

1. Stream and Velocity Potential Functions for Three-Dimensional Flow. The functions φ and ψ as hitherto employed and arising out of the development of some function of $(x + iy)$ are not applicable to fields of flow in three-dimensional space. The definition of φ, however, remains, as in Division A VII 1—a function whose derivative along any direction in space will give the velocity in that direction. For a suitable definition of the function ψ we may consider that a flow in space is made up of a series of surfaces or sheets of flow, while these, in turn, are made up of a series of lines of flow—stream-lines in space. It is clear that the fluid flowing between two such surfaces in one part of the field will always remain between these same surfaces throughout the extent of the flow. The function ψ may then be defined as the total rate of flow between some one such surface taken as datum and any other surface identified by a special value of the function.

This is entirely parallel with the definition for two-dimensional flow in Division A VIII 2. The particular form of surface to be employed will depend on the character of the flow. For the case of a field of flow symmetrical about a straight line, the relation between the velocities in the field and the function ψ are especially simple in form.

Thus in Fig. 55 let the flow be symmetrical about the axis of X and let AP and BQ be two stream-lines in an axial plane which we may take as the plane of XY.

Then A_1P_1 and B_1Q_1 will represent the corresponding lines on the opposite side of the axis. There will be, therefore, surfaces of flow made up of the assemblage of these lines all around the axis and forming two surfaces of revolution with XX as axis and AP and BQ as generatrices.

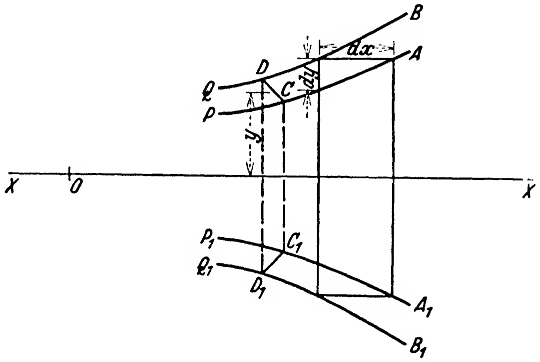

Fig. 55.

Between these two surfaces there will be a sheet of flow, the axial section of which is shown by the diagram.

Then as the function ψ has been defined, there will be a value ψ_1 for the surface APA_1P_1 and a value ψ_2 for the surface BQB_1Q_1, and the flow in the sheet between these surfaces will be $\psi_2 - \psi_1$. Now let AP and BQ be very near together. The difference $\psi_2 - \psi_1$ will then become $d\psi$. Let CD be any line drawn from AP to BQ and then let CD be carried around X as an axis, thus generating a continuous partition across the sheet of flow, CDD_1C_1.

Let y be the ordinate of CD. Then the area of this partition will be $2\pi y CD$. Let s denote in general the direction of any such line CD. Then we may denote CD by ds.

Let V denote the velocity $\perp$ to CD. We shall then have

$$V \cdot 2\pi y\, ds = d\psi$$

whence

$$V = \frac{1}{2\pi y}\frac{\partial \psi}{\partial s} \tag{1.1}$$

This gives then a general formula for finding at any point in such a field and in terms of the function ψ, the velocity in any specified direction in the plane XY. We have simply to take the derivative of ψ in a direction at $+90^0$ with the specified direction and divide the result by $2\pi y$, the circumference of the sheet of flow at that point.

If now we take CD or s, parallel first to Y and then to X, we shall have, for the velocities parallel to X and to Y,

$$u = \frac{1}{2\pi y}\frac{\partial \psi}{\partial y}$$

$$v = -\frac{1}{2\pi y}\frac{\partial \psi}{\partial x}$$

The negative sign develops from the assumption that ψ is increasing from AP to BQ and for such $+d\psi$, dx is negative as shown.

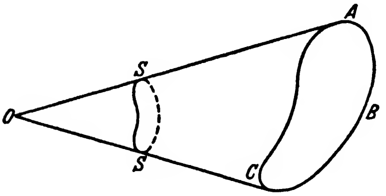

Fig. 56.

While these results are expressed in terms of x and y as referring to the plane of XY, it is clear that the values are the same in all planes through OX and that these relations are thus general throughout the field of any such flow symmetrical about a line which may be taken as the axis of X.

Combining these values of u and v with those resulting from the use of φ we have

$$\left.\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{1}{2\pi y}\frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \varphi}{\partial y} = -\frac{1}{2\pi y}\frac{\partial \psi}{\partial x} \end{aligned}\right\} \qquad (1.2)$$

It should be remembered that the above values, so far as φ is concerned are general and apply in any field while those in terms of ψ apply only where the field is symmetrical about the axis of X.

The relation between φ and ψ at any point in such a symmetrical field may be put in a single equation as follows:

$$\frac{\partial \varphi}{\partial l} = \frac{1}{2\pi y}\frac{\partial \psi}{\partial n} \qquad (1.3)$$

where l is any direction in the axial plane and n is the direction at $+ 90^0$. It results that in such a field, ψ may be derived from φ or *vice versa*. Thus if φ is known, we have only to find $\partial\varphi/\partial l$, multiply by $2\pi y$, express in terms of n and integrate to find ψ. If ψ is known, we find $\partial\psi/\partial n$, divide by $2\pi y$, express in terms of l and integrate to find φ. Naturally the operative program may be much simplified by a proper selection of the direction l or n.

2. Sources and Sinks—Three-Dimensional Field. We may imagine at a point in three-dimensional space, fluid issuing at a uniform rate and flowing in all directions. Assume a sphere of unit radius about this point as center. The surface of this sphere will be 4π. Let m be the total rate of flow issuing from the source. Then $m/4\pi$ is the rate of flow per unit area of surface on the sphere.

Let ABC Fig. 56 be any closed area in space and O a source. Assume lines drawn from O touching all points of the boundary of ABC

thus forming a solid cone with apex at O. We remember that the measure of the solid angle at O subtended by the area $A\,B\,C$ is taken as the area SS cut out of the surface of a sphere of unit radius about O as center. The total area of the sphere is $4\,\pi$. Hence the total solid angle about a point in space is $4\,\pi$. If then ω is the area of any closed figure on the surface of this unit sphere, the solid angle subtended at the center is measured by ω.

It follows that the rate of flow from the source per unit of solid angle will be $m/4\pi$.

For a sink in space, the relations are the same with, simply, a negative sign for m.

For the case of a single source, the fluid issuing into an indefinite field at rest, will obviously flow away radially and equally in all directions in space and the velocity at any point in space distant r from the source will be that due to a total flow m across the surface of a sphere of radius r. This will give for such velocity the value

$$V = \frac{m}{4\,\pi\,r^2}$$

3. Stream and Velocity Potential Functions for a Source or Sink—Three-Dimensional Space. The flow will here be symmetrical about an axis X and the formulae of **1** will apply.

As we have seen, the fluid, in such a case will flow away radially and equally in all directions in space. The surfaces of flow may then be taken as formed by a series of cones as in Fig. 57, having a common axis OX and progressive half angles at the apex θ_1, θ_2, θ_3, etc. The datum in such case will naturally be the limit of such a series of cones which is the axis of X.

The velocity at any point in space distant r from the source will be $m/4\,\pi\,r^2$. But if there is a potential φ, this velocity will be represented by $\partial\varphi/\partial r$. Hence we have $\dfrac{\partial\varphi}{\partial r} = \dfrac{m}{4\,\pi\,r^2}$

This gives, on integration, $\qquad \varphi = -\dfrac{m}{4\,\pi\,r} \qquad$ (3.1)

In order to find ψ we may express the same velocity in terms of this function. To this end we must take the derivative of ψ in a direction at $+\,90^0$ with the radius r [see (1.3)] and then divide by $2\,\pi\,y = 2\,\pi\,r\,\sin\theta$

This gives the equation $\qquad \dfrac{m}{4\,\pi\,r^2} = \dfrac{1}{2\,\pi\,r\,\sin\theta}\,\dfrac{\partial\psi}{r\,\partial\theta}$

whence $\qquad d\psi = \dfrac{m}{2}\sin\theta\,d\theta$

and $\qquad \psi = -\dfrac{m}{2}\cos\theta\Big]_0^\theta$

or $\qquad \psi = \dfrac{m}{2}\,(1 - \cos\theta) \qquad$ (3.2)

This should then be the total flow radially outward through a cone of half angle θ at the apex. This may be checked as follows. It is well known that the area cut out of a sphere of radius r and a cone of half angle θ is measured by $2\pi r^2 (1 - \cos\theta)$. The flow across such an area will be given by multiplying the area by the velocity $m/4\pi r^2$. This gives the result in (3.2).

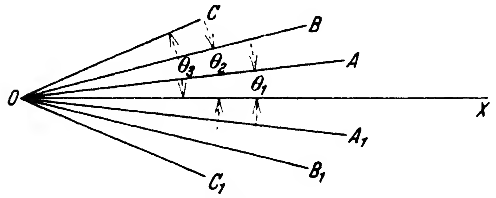

Fig. 57.

Since the flow is symmetrical about any axis such as X, we have a complete picture of the flow by taking any section through the axis as in Fig. 57. The stream-lines in this section will be given by $\psi = const.$

$$\cos\theta = 1 - \frac{2\psi_1}{m}$$

or

$$\theta = \text{const.}$$

thus giving lines such as OA and OA_1 indefinitely extended.

For a sink the results are the same, with only a negative sign for m.

4. Combination of a Source and Sink of Equal Strength—Three-Dimensional Field. Let the source be at A and the sink at B (see Fig. 58).

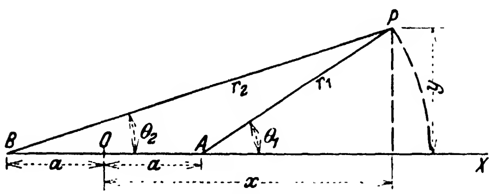

Fig. 58.

Then at a point P,

$$\varphi_A = -\frac{m}{4\pi r_1}$$

$$\varphi_B = \frac{m}{4\pi r_2}$$

whence summing:

$$\varphi = \frac{m}{4\pi}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{4.1}$$

From this equation, equipotential lines for a section through X may be computed or constructed.

Again for the function ψ, we have

$$\psi_A = \frac{m(1 - \cos\theta_1)}{2}$$

$$\psi_B = \frac{-m(1 - \cos\theta_2)}{2}$$

whence

$$\psi = \frac{m}{2}(\cos\theta_2 - \cos\theta_1) \tag{4.2}$$

Similarly here, stream-lines for an axial section may be computed or constructed from this equation for ψ (see Fig. 59).

5. Combination of Two Sources of Equal Strength—Three-Dimensional Field. In Fig. 60 let the sources be at A and B. Then it is readily seen that we shall have

$$\left.\begin{aligned}\varphi &= -\frac{m}{4\pi}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\\ \psi &= m\left(1-\frac{\cos\theta_1+\cos\theta_2}{2}\right)\end{aligned}\right\} \qquad (5.1)$$

Such a field is shown in Fig. 60. In the same manner as for flow in two dimensions, we may here substitute a rigid smooth surface for any of the flow surfaces without in any way changing the conditions

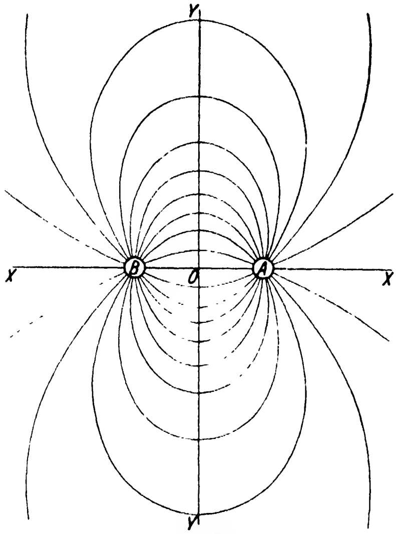

Fig. 59.

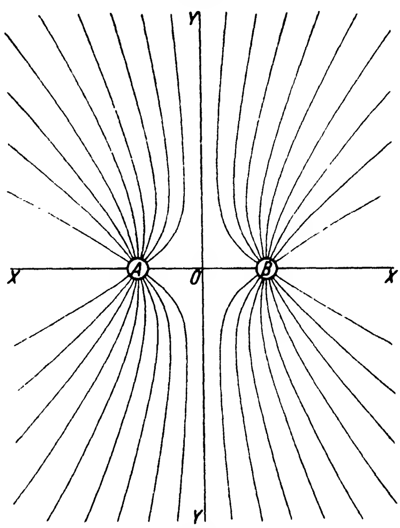

Fig. 60.

on either side of such surface, or we may suppress the flow on either side without affecting that on the other. Thus in Fig. 60 we may consider the part of the diagram on the right of $Y\,Y$, for example, as representing a source near an indefinite plane surface. The form of the stream-line flow is then given as in the diagram.

For two sinks of equal strength, the diagram is the same with the direction of flow reversed.

6. Combinations of Sources and Sinks of Unequal Strengths—Three-Dimensional Field. By the proper combination of the suitable elements, we find directly as follows:

Source and Sink, Unequal Strength

$$\left.\begin{aligned}\varphi &= \frac{1}{4\pi}\left(\frac{m_2}{r_2}-\frac{m_1}{r_1}\right)\\ \psi &= \frac{m_1(1-\cos\theta_1)-m_2(1-\cos\theta_2)}{2}\end{aligned}\right\} \qquad (6.1)$$

Two Sources, Unequal Strength

$$\left.\begin{aligned} \varphi &= -\frac{1}{4\pi}\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) \\ \psi &= \frac{m_1(1-\cos\theta_1) + m_2(1-\cos\theta_2)}{2} \end{aligned}\right| \tag{6.2}$$

7. Doublet in Three-Dimensional Space. Similar to the case of IV **10** we may assume the distance between the source and sink to decrease and the strength m to correspondingly increase, the product remaining constant and equal to say M. We have now to find the functions φ and ψ for such a doublet.

This is evidently the case of 4 when a becomes indefinitely small. Referring again to Fig. 58, take the midpoint O as origin and coordinates x and y for the point P as shown. Then,

$$\varphi = \frac{m}{4\pi}\left[\frac{1}{\sqrt{(x+a)^2+y^2}} - \frac{1}{\sqrt{(x-a)^2+y^2}}\right]$$

If then we expand $(x+a)^2$ and $(x-a)^2$, put $x^2+y^2=r^2$ reject a^2 and take the square root considering a small, we shall have

$$\varphi = \frac{m}{4\pi}\left[\frac{1}{r+\frac{ax}{r}} - \frac{1}{r-\frac{ax}{r}}\right]$$

Combining these and reducing we find

$$\varphi = -\frac{m}{4\pi}\,\frac{2ax}{r^3}$$

Then putting $2am = M$ and $x/r = \cos\theta$ we have finally

$$\varphi = -\frac{M}{4\pi}\,\frac{\cos\theta}{r^2} \tag{7.1}$$

Similarly for ψ we have

$$\psi = \frac{m}{2}\left(\frac{x+a}{r_2} - \frac{x-a}{r_1}\right)$$

Taking the same values for r_2 and r_1 as for φ and reducing we find,

$$\psi = \frac{m}{2}\left[\frac{2a(r^2-x^2)}{r^3}\right] = \frac{m}{2}\left(\frac{2ay^2}{r^3}\right)$$

Then putting $2am = M$ and $y = r\sin\theta$ we have:

$$\psi = \frac{M}{2}\,\frac{\sin^2\theta}{r} \tag{7.2}$$

An axial section of such a field is shown in Fig. 61.

As an example of the derivation of φ from ψ or *vice versa*, as discussed in **1**, assume φ known and ψ desired. Then

$$\frac{\partial\varphi}{r\,\partial\theta} = \frac{M}{4\pi}\,\frac{\sin\theta}{r^3} = -\frac{1}{2\pi r\sin\theta}\,\frac{\partial\psi}{\partial r}$$

Here $\partial\varphi/r\,\partial\theta$ gives the velocity $\perp$ to r. To obtain this velocity from ψ we must take the derivative in a direction at $+90^0$ or along $-r$. Then

with $y = r \sin\theta$ we have the expression on the right. Reducing and integrating relative to r, we find

$$\psi = \frac{M}{2} \frac{\sin^2\theta}{r} \text{ as in (7.2).}$$

8. Combinations of Sources and Sinks Along a Straight Line—Three-Dimensional Space. This problem is in every way similar to the

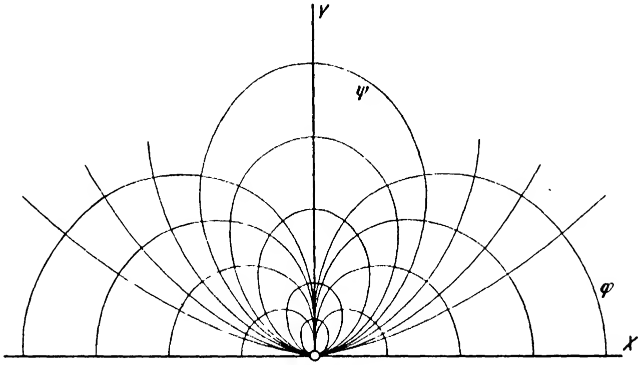

Fig. 61.

corresponding problem in a plane (see IV **11**). The field will be symmetrical about the line of source—sink flow.

For an axial section through the field of stream-line flow, we have then simply to follow the same general procedure as in IV **11** using

the function $$\psi = \frac{m}{2}(1 - \cos\theta) \quad \text{[see (3.2)]}.$$

For the field of equipotentials, the procedure is the same with the substitution of the proper function as in **3**.

If the elements along the line are doublets instead of individual sources or sinks, the procedure is the same with the use of the proper functions as in **7**.

9. Field for a Continuous Distribution of Sources and Sinks Along a Straight Line—Three-Dimensional Space. This again is in every way similar to the corresponding case in two dimensions (see IV **12**). Here as with IV **11** and IV **12** the procedure only differs from that of the preceding section by the need of some method of integration for the continuous distribution of source or sink strength.

In Fig. 62 let $ABCDE$ denote the distribution, Q any point on the line and P any point in the field. Then with the functions φ and ψ as in **3** and with the procedure of IV **12** we shall have:

$$\varphi = -\frac{1}{4\pi}\int \frac{f(x)\,dx}{r}$$

$$\psi = \frac{1}{2}\int f(x)(1 - \cos\theta)\,dx$$

Then by the aid of auxiliary diagrams, as in IV 12, a regular system of stream-lines or equipotentials may be developed as desired. These will, of course, belong to an axial section. As in other cases, such a section revolved about the axis XX_1 will generate the sheets of flow which may be conceived of as making up the field in space of three dimensions.

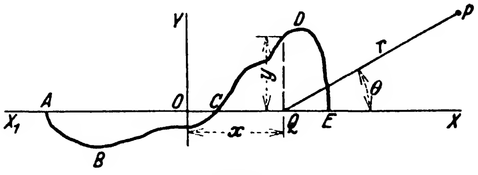

Fig. 62.

Reference may here be made to Division C V 5, VI 1, 4, VII 6 where some discussion is given of cases where the inverse procedure may be realized—the determination of a distribution of sources and sinks, or of doublets, which will give a field of known velocity potential.

10. Combination of Source with Uniform Flow: Three-Dimensional Space. The functions φ and ψ for a source are as given in **3**.

The function φ for an indefinite flow is the same as for two-dimensional flow.

The function ψ, however, is not the same as for two-dimensional flow and must be derived in accordance with the definition of **1**. As there defined, the surfaces of flow will be the surfaces of cylinders about an axis X and with radius y, and the function ψ will be the total flow within such a cylinder.

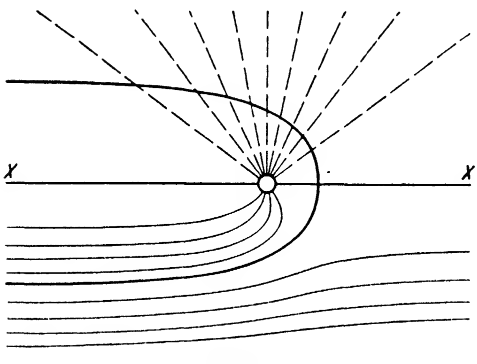

Fig. 63.

We shall have, then, these two functions

$$\varphi = -Ux$$
$$\psi = -\pi U y^2$$

It will be seen that the formulae of **1** applied to these functions will give the correct values of the velocity for the indefinite uniform flow.

The combination will give therefore,

$$\varphi = -Ux - \frac{m}{4\pi r} \tag{10.1}$$

$$\psi = -\pi U y^2 + \frac{m}{2}(1 - \cos\theta) \tag{10.2}$$

The flow will obviously be symmetrical about the axis of x. Putting $\psi = 0$ we find
$$y = \sqrt{\frac{m}{\pi U}} \sin\frac{1}{2}\theta$$
This will be maximum for $\theta = 180^0$ and here $y = \sqrt{m/\pi U}$.

The maximum radius of the cross section of the surface, thus separating the two flows will, therefore, be at ∞ and will have this value.

Call this radius R, then $y = R \sin \frac{1}{2} \theta$

Also $y = r \sin \theta$ and $x = r \cos \theta$, whence we find,

$$r = \frac{R}{2 \cos \frac{1}{2} \theta}$$

$$x = \frac{R \cos \theta}{2 \cos \frac{1}{2} \theta}$$

From these values the form of the meridian section is readily found, as shown in Fig. 63.

Also from (10.1) we find

At $x = -\infty$, $u = -U$, $v = 0$

At $x = R/2$, $y = 0$, $u = 0$ and $v = 0$ (Pt. of stagnation)

11. Combination of Space Doublet with Indefinite Flow Parallel to the Axis of X—Indefinite Flow About a Sphere. The functions φ and ψ for a space doublet are as given in **7**. The functions φ and ψ for the indefinite flow are as in **10**. The combination will give therefore,

$$\varphi = -\frac{M}{4\pi} \frac{\cos \theta}{r^2} - U x = -x \left[\frac{M}{4\pi r^3} + U \right] \qquad (11.1)$$

$$\psi = \frac{M}{2} \frac{\sin^2 \theta}{r} - \pi U y^2$$

or

$$\psi = y^2 \left[\frac{M}{2 r^3} - \pi U \right] \qquad (11.2)$$

Putting $\psi = 0$ we have

$$r = \sqrt[3]{\frac{M}{2\pi U}} \qquad (11.3)$$

But M and U are constant and hence this will give a circle about O as center. But this same circle will exist for every axial section through the field of flow and hence the surface, of which this is only one axial section, will be a sphere with radius as in (11.3). It is also of interest to note that the volume of this sphere will be $2M/3U$.

Denoting this special radius by a we have

$$a = \sqrt[3]{\frac{M}{2\pi U}} \quad \text{and} \quad M = 2\pi a^3 U$$

This gives

$$\varphi = -U x \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \right] \qquad (11.4)$$

$$\psi = -\pi U y^2 \left[1 - \left(\frac{a}{r} \right)^3 \right] \qquad (11.5)$$

The surface of this sphere will, then, mark out a surface of separation between the fluid issuing from the doublet and that flowing in the external field. The case is in every way similar to that of V 2 for flow

in two directions. The field of flow outside this sphere will be, therefore, the field for an indefinite rectilinear flow about a sphere. The stream-lines for an axial section of this field are shown in Fig. 64.

If we take the derivative of (11.1) relative to x, putting $\theta = 0$, we shall have the velocity along the axis of x; and if we then put $r = a$ and substitute $M = 2\pi a^3 U$ we shall have zero for the velocity at the point of stagnation, where X meets the sphere—all as we should expect. If again in (1.1) we put $x = r\cos\theta$, find the derivative $\partial\varphi/r\partial\theta$, put $r = a$ and substitute $M = 2\pi a^3 U$, we shall have the velocity along the surface at points on the sphere in the form $(3/2)$ $(U\sin\theta)$. This again gives zero for $\theta = 0$ while for $\theta = 90^0$ we have $(3/2)\,U$, the maximum velocity across the equator of the sphere.

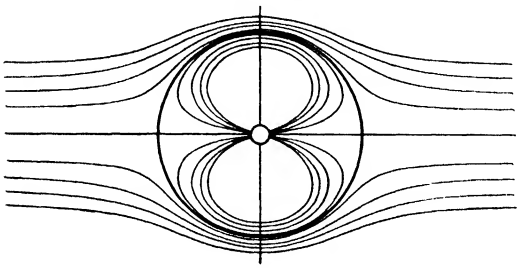

Fig. 64.

12. Field of Flow for a Sphere Moving in a Straight Line in an Indefinite Fluid Field. The functions φ and ψ for this case are derived from those of **11** in exactly the same manner as in the case of two-dimensional flow (see VII 4). But it is evident that this will simply reproduce the functions for the space doublet as in **7**. This again is parallel to the case of flow in two dimensions, where the functions for a circle moving in an indefinite field are the same as for a plane doublet.

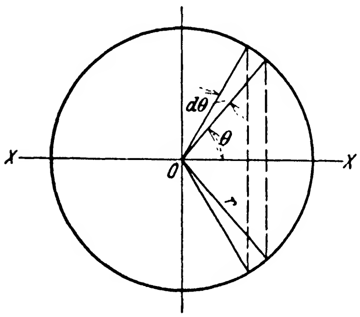

Fig. 65.

To find the energy in this field, we use the general formula of Division A IX (2.3)

$$E = -\frac{\varrho}{2}\int \varphi\,\frac{\partial\varphi}{\partial n}\,d\,S$$

Here we have:

$$\varphi = -\frac{M}{4\pi}\,\frac{\cos\theta}{r^2}$$

Also here, $\partial\varphi/\partial n = \partial\varphi/\partial r$ or

$$\frac{\partial\varphi}{\partial n} = \frac{M}{2\pi}\,\frac{\cos\theta}{r^3}$$

For the element of area take the circular band swept out on the surface of the sphere by the arc of length $r\,d\,\theta$ as in Fig. 65. This will give:

$$dS = r\,d\,\theta\cdot 2\pi r\sin\theta = 2\pi r^2\sin\theta\,d\,\theta$$

Multiplying these together, integrating for θ between limits of 0 and π, putting a for r and $2\pi a^3 U$ for M (see **11**), we find

$$E = \frac{\varrho}{2} \cdot \frac{2}{3} \pi a^3 U^2$$

This is the energy of a volume equal to one-half that of the sphere, of density ϱ and moving with a velocity U.

The stream-lines for an axial section of this field are shown in Fig. 66.

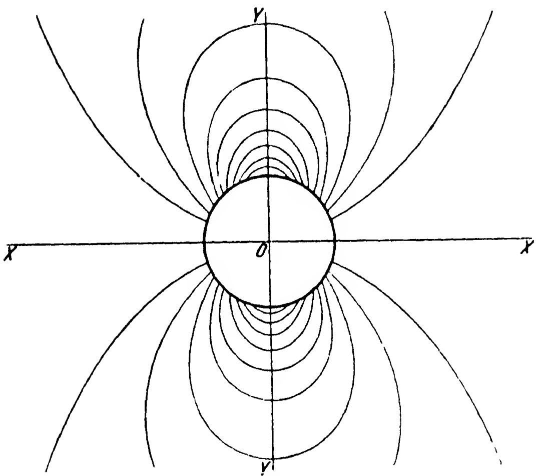

Fig. 66.

13. Rectilinear Flow with the Source and Sink Distributions of 8 and 9. The procedure in this case, again, is parallel with that of V 3. The values of φ or ψ are determined as in 8 or 9 and to these must then be added the value for the rectilinear flow as in **10**. This will give the total value at a given point, and such values are then to be combined and treated generally in accordance with the same procedure as in IV **12**, V **3**.

14. Any Field of Flow as the Result of a Distributed System of Sources and Sinks or of Doublets. The introduction of a small pair (source and sink) into a field of moving fluid will have the same result as the introduction of a small solid body of form depending on the characteristics of the pair. That is, the original fluid will move as though it were flowing about this small solid body. We may then conceive of an indefinite number of these small elements, grouped in such manner and with such spacing as to constitute a closed surface of any extent and any form whatever. We may also, if we choose, consider the interior of such volume packed full of these small elements, each resulting from a source and sink pair.

The assemblage of such elements will then give an external field of flow the same as for a solid body of the same size and shape as the external boundary. In this manner it is possible to conceive of any such field of flow (non-cyclic in character) as formed by the combination of an indefinite number of small pairs disposed over a closed surface representing the form and dimensions of the body around which the flow is assumed to take place.

The same general principle holds, of course, for two-dimensional flow where we have only to dispose our small elements around the contour of the form in question, or, if we choose, pack them in so as to completely fill the area within such contour.

We have thus far spoken of the small element as due to a source and sink pair, but without reference to the distance between them. Different distances will, of course, give different forms of element, other things being the same. It is clear, however, that the same over all result may be realized by the use of small doublets instead of unspecified pairs, and all that has been said above with regard to such pairs will apply equally well to the use of doublets as small elements.

While it is not, in general, practicable to use such distributions in space for the solution of physical problems, the concept is of interest as furnishing an ideal method whereby the most complex results may be conceived of as having a definite physical background.

Reference may also be made to Division C V 5, VI 1, 4, VII 6, for some discussion of cases where a distribution of doublets along a line or over a surface may be determined in such way as to give a field of known velocity potential.

CHAPTER X

AEROSTATICS: STRUCTURE OF THE ATMOSPHERE

1. Buoyancy. As noted in I 1 an ideal fluid is conceived of as composed of small elements or particles having mass and moving freely and without shearing stress, one relative to another. We must also assume the existence of a gravity field as a result of which the pressure at any point in the fluid is measured by the action of gravity on the particles lying between such point and the outer boundary of the fluid. Thus with the atmosphere surrounding the earth, the pressure at any point (measured as the force on unit area) is equal to the weight of a column of air of unit area base and extending from the given point to the upper limits of the atmosphere. From the absence of shear it also results that such pressure at any point will act equally in all directions, and thus the pressure on the surface of any body immersed in the fluid will be $\perp$ to such surface at each point.

The total force reaction between a solid body and a fluid in which it is immersed develops out of this fundamental property of the ideal fluid.

For geometrical reference we take axes X and Y horizontal and Z vertical or in the direction of the gravity field. In Fig. 67 let $ADBC$ denote a body of any form immersed in a fluid. Let AB denote an element of the body formed with AB in the direction of Z and with section $dx\,dy$ while CD denotes a similar element with CD in the direction of X and with section $dy\,dz$.

In the field as assumed, surfaces of equal pressure will be parallel to the plane XY and $\perp$ to Z. Hence the pressures at C and D will be the same. Denote this pressure by p, the angle between the normal to the surface at C and the direction of X by θ, and the area of the element of the surface by dS. Then the resultant pressure inward at C along the direction of X will be

$$dP = p\,dS\cos\theta$$

But $dS\cos\theta = dy\,dz$ and hence

$$dP = p\,dy\,dz \text{ directed from } C \text{ toward } D.$$

But by the same reasoning, the resultant pressure at D along X will be likewise $p\,dy\,dz$ directed from D toward C. Hence these two resultants will cancel and the net X resultant on the elements of surface at C and D will be zero. The same will be true for every other like element of volume and hence for the body as a whole. The same reasoning will hold for the resultant pressures in the direction of Y and since the directions of X and Y are in no wise limited with regard to the form of the body, the result as stated will hold for any and all directions in the plane XY.

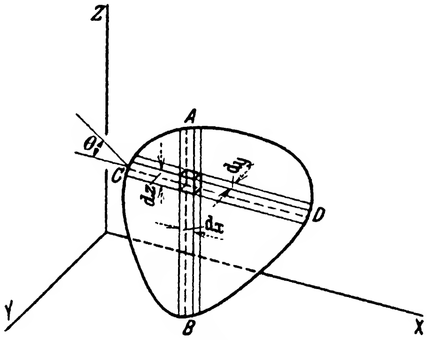

Fig. 67.

We thus reach the important result that in a fluid field as assumed, the total resultant pressure in any direction parallel to the planes of equal pressure (or $\perp$ to the direction of gravity) will be zero and the body will remain in equilibrium so far as such pressures are concerned.

Turning now to the pressures in the direction of Z, let p_1 denote the pressure at the element A and p_2 that at B. Then following the same reasoning as for CD the resultant pressure at A along Z will be $p_1\,dx\,dy$ directed from A to B and that at B will be $p_2\,dx\,dy$ directed from B to A. Hence the net resultant Z pressure on the element AB will be

$$\triangle P = (p_2 - p_1)\,dx\,dy$$

directed from B to A.

But in a gravity field we have

$$(p_2 - p_1) = \varrho \triangle z$$

when $\qquad \varrho =$ density of the fluid

$\qquad \triangle z =$ length of element AB.

Strictly, ϱ is the mean value of the density between the levels of A and B. In most practical problems, however, the change in density over a vertical distance corresponding to the dimensions of any given

body is so small that ϱ may be taken as constant at the value corresponding to the general location of the body in the fluid.

We have thus $\quad \triangle P_z = \varrho \triangle z\, dx\, dy = \varrho \triangle V$

where $\triangle V$ is the volume of the element AB.

The same will be true for any and all other such elements and hence for the sum, the volume as a whole. Hence finally

$$P_z = \varrho V$$

acting along the direction of decreasing density.

Or in words, the vertical resultant of the surface pressures on the body is measured by the product of the density of the fluid and the volume of the body, and this is equal to the weight of a volume of the fluid equal to that displaced by the body.

This result has been reached by a simple application of elementary mechanics. It may be of interest to deduce the same result without reference to the action on the geometrical elements of the body.

We may picture a geometrical volume of the given size and form inclosed by an envelop without thickness or mass and filled with the fluid in which the body is assumed to be immersed. This is, in effect merely a method of isolating, in thought, a volume of the fluid of the size and form of the body in question. Obviously the relation between the fluid thus isolated and the remainder of the fluid will be one of equilibrium. But in a gravity field the reaction of the isolated fluid upon the remainder of the fluid will be measured by the weight of the isolated fluid. Hence the equal and opposite reaction of the remainder of the fluid upon the isolated volume will have the same measure. But such action can only be effected at the surface of separation between the two parts of the fluid as a whole. But the action of the outlying fluid on the surface of separation must be the same regardless of what is inside this surface, and thus it will be the same if, instead of a portion of fluid isolated by a geometrical boundary surface, we suppose the space occupied by any body whatever, having the same size and form. Hence the action of the outlying fluid on such a body will equal the weight of a volume of fluid equal to that of the body and this, as before, gives

$$P_z = \varrho V$$

2. Center of the System of Surface Pressures. In **1** it is shown that the element of the vertical resultant of the system of surface pressures is equal to the weight of a volume of the fluid equal to that of the element itself. The entire system of vertical pressures will be the same, therefore (with direction reversed), as that represented by gravity acting on the given volume filled with the fluid. The center of such a system of forces must, therefore, be the same as that of a gravity system acting on a body of the given form and of density ϱ (assumed here uniform throughout the given volume). But this will be the center of gravity of

a homogeneous body of the given form, or otherwise, the center of figure of the given volume.

3. Structure of the Atmosphere: Standard Atmosphere. The atmosphere is composed chiefly of oxygen and nitrogen with small, and, for our present purpose, unimportant fractions of other gases such as carbon dioxide, argon, hydrogen, etc., and with a small and varying percentage of water vapor.

The proportions of oxygen and nitrogen by weight are nearly 23 to 75.6 and by volume 20.9 to 78.1, the remainder being made up by the other constituents.

For aerodynamic purposes, the chief interest in the atmosphere lies in its various physical properties (pressure, temperature, density) and in their variation with altitude.

Observations with sounding balloons have shown that for altitudes between a lower limit of 10,000 to 12,000 meters (32,800 to 39,360 feet) and an upper limit of about 20,000 meters (65,600 feet) the temperature remains nearly constant at about -55° C (-67° F). This is known as the isothermal layer.

Extended measurements on the decrease of temperature with altitude between the limits of sea level and the lower boundary of the isothermal layer, have also justified, for practical purposes, the use of Toussaint's equation.

$$T = T_0 - a\,Z \tag{3.1}$$

where T = temperature at altitude Z

T_0 = temperature at sea level

a = constant temperature gradient.

Extended investigations have furthermore given for this temperature gradient the value .0065 degree Centigrade per meter of elevation or .003566 degree Fahrenheit per foot of elevation. This gives for the law of temperature variation with altitude,

$$T = T_0 - .0065Z \text{ (Centigrade and meters)} \tag{3.2}$$

$$T = T_0 - .003566Z \text{ (Fahrenheit and feet)} \tag{3.3}$$

It must be understood that this linear law is only to be applied below the limit of the isothermal layer. These values of the temperature gradient together with the assumed value of the temperature of the isothermal layer will give the upper limit of application of this formula. Thus for the Centigrade scale putting $T = 218^\circ$ abs. and $T_0 = 288^\circ$ (15° on the common scale) we find $Z = 10769$ meters. Similarly the Fahrenheit scale gives $Z = 35332$ feet. For purposes of computation, therefore, these limits are assumed for the application of this law of temperature variation with altitude.

In addition to this assumed law of temperature variation, certain further assumptions are needed in order to define what is known as the standard atmosphere, and in order to bring its physical properties

within the range of ready mathematical determination. These further assumptions are as follows:

(1) The air is dry.

(2) The air behaves as a perfect gas following the laws of Charles and of Boyle, *viz.*

$$p = Rg\,\varrho\,T \tag{3.4}$$

$$\frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0}\right)\left(\frac{T}{T_0}\right) \tag{3.5}$$

(3) Gravity is taken as constant at all altitudes or otherwise, within the range of altitudes involved in aeronautic problems, the variation of gravity is ignored.

In other words, the standard atmosphere is considered as a dry perfect gas, in a constant gravity field, with an isothermal layer at an altitude of about 11,000 meters, and within this altitude, following the linear law of variation of temperature with altitude as given in (3.1).

4. Derivation of Formulae. With the assumptions made, it is evident that the difference in pressure between any two levels must be due to the weight of a column of air of unit cross section extending between these two levels. This gives immediately the following differential equation:

$$dp = -g\,\varrho\,dZ \tag{4.1}$$

Then substituting from (3.4) we have

$$-\int \frac{dp}{p} = \frac{1}{R}\int \frac{dZ}{T} \tag{4.2}$$

We now define a mean temperature T_m by the equation

$$T_m = \frac{\int_0^Z dZ}{\int_0^Z \frac{dZ}{T}} = \frac{\int_0^Z dZ}{\int_0^Z \frac{dZ}{T_0 - aZ}} = \frac{aZ}{log\,(T_0/T)} \tag{4.3}$$ [1]

Then combining (4.2) and the first members of (4.3) we find

$$-\int_{p_0}^{p} \frac{dp}{p} = \frac{Z}{R\,T_m}$$

or

$$Z = R\,T_m log\left(\frac{p_0}{p}\right) \tag{4.4}$$

But from (3.4) $R = p_0/g\,\varrho_0\,T_0$ and substituting, we have

$$log\left(\frac{p_0}{p}\right) = \frac{\varrho_0\,g\,Z}{p_0}\left(\frac{T_0}{T_m}\right) \tag{4.5}$$

This will serve to determine the pressure p for any given altitude Z and temperature T and this with (3.5) will give the density ϱ.

[1] It will be noted that *log* without subscript always implies log_e.

Thus by a suitable use of these equations, all desired relations may be established between the various characteristics: pressure, temperature, density, altitude.

However, above the lower limit of the isothermal layer, (4.3) cannot be used as the defining equation for T_m. For altitudes above this limit use may be made of the form

$$T_m = \frac{Z}{\frac{Z_1}{T_{m1}} + \frac{Z - Z_1}{T_1}} \tag{4.6}$$

where Z is the actual level

Z_1 the isothermal level

T_{m1} the value of T_m at Z_1 as given by (4.3)

T_1 the isothermal temperature.

This gives numerically

$$T_m = \frac{Z}{42.839 + \frac{(Z - 10769)}{218}} \quad \text{(metric)}, \ Z > 10769 \text{ m.}$$

$$T_m = \frac{Z}{78.051 + \frac{Z - 35332}{392.4}} \quad \text{(English)}, \ Z > 35332 \text{ ft.}$$

For points above the isothermal layer, the following relation between the pressure ratio and the density ratio will be of use. Denoting conditions at the isothermal level by sub 1, we have:

$$\frac{p_0}{p} = \frac{p_0}{p_1}\frac{p_1}{p}$$

But for isothermal conditions $p_1/p = \varrho_1/\varrho$. Hence we have:

$$\frac{p_0}{p} = \frac{p_0}{p_1}\frac{\varrho_1}{\varrho} = \left(\frac{p_0}{p_1}\frac{\varrho_1}{\varrho_0}\right)\frac{\varrho_0}{\varrho}.$$

Or putting in numerical values we have

$$\frac{p_0}{p} = 1.3211\frac{\varrho_0}{\varrho} \tag{4.7}$$

That is, above the level of the isothermal layer, the relation between the pressure ratio and the density ratio retains the value for that level. This will give the over all pressure ratio in terms of the over all density ratio or *vice versa*.

If, then we take the value of T_m in (4.6) as the numerical value (English) for Z_1/T_{m1} and substitute in (4.4), we shall have:

$$(Z - Z_1) = T_1\left(R \log\frac{p_0}{p} - 78.051\right) \tag{4.8}$$

This equation may be used to find the elevation above the isothermal layer where p_0/p is given. If ϱ_0/ϱ is given, find p_0/p by (4.7) and proceed with (4.8).

If the quantities are in metric units, 42.839 must be used instead of 78.051.

Or otherwise in terms of $p_1/p = \varrho_1/\varrho$, we have, for the elevation above the isothermal level:

$$(Z_1 - Z) = R\, T_1 \log \left(\frac{p_1}{p}\right) \tag{4.9}$$

Below the isothermal level there exist certain interesting and useful relations which are readily derived from the preceding basic equations.

From (4.2) and (3.1) we have

$$-\int \frac{dp}{p} = \int \frac{dZ}{R(T_0 - aZ)}$$

whence

$$a\, R \log \left(\frac{p}{p_0}\right) = \log \left(\frac{T}{T_0}\right)$$

or

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{aR}$$

The quantity aR is non-dimensional and has the value .19026. We may therefore put

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{.19}$$

or

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{5.256}$$

Combining these various relations suitably, the following formulae are readily established.

$$\left(\frac{\varrho}{\varrho_0}\right) = \left(\frac{p}{p_0}\right)^{.81}$$

$$\left(\frac{p}{p_0}\right) = \left(\frac{\varrho}{\varrho_0}\right)^{1.235}$$

$$\left(\frac{T}{T_0}\right) = \left(\frac{\varrho}{\varrho_0}\right)^{0.235}$$

$$\left(\frac{\varrho}{\varrho_0}\right) = \left(\frac{T}{T_0}\right)^{4.256}$$

$$\left(\frac{\varrho}{\varrho_0}\right)^{0.235} = \left(1 - \frac{a}{T_0} Z\right)$$

$$\left(\frac{\varrho}{\varrho_0}\right) = \left(1 - \frac{a}{T_0} Z\right)^{4.256}$$

$$\left(\frac{p}{p_0}\right)^{.19} = \left(1 - \frac{a}{T_0} Z\right)$$

$$\left(\frac{p}{p_0}\right) = \left(1 - \frac{a}{T_0} Z\right)^{5.256}$$

It must be remembered that these formulae only apply below the level of the isothermal layer.

For use in connection with these various equations the National Advisory Committee for Aeronautics (U. S. A.) has recommended the following standard values.

Pressure	p_0 = 760 mm	= 29.921 in.
Temperature	t_0 = 15° C	= 59° F.
Absolute Temperature	T_0 = 288° C	= 518.4° F.
Specific Weight	$g\varrho_0$ = 1.2255 kg/m³	= 0.07651 lb/ft³.
Gravity	g = 9.80665 m/sec²	= 32.1740 ft/sec².
Density	ϱ_0 = 0.12498 kg/m/sec	= 0.002378 lb/ft/sec.
Temp. Gradient	a = 0.0065° C/m	= 0.003566° F/ft.
Gas Constant	R = 29.2708 metric	= 53.33 English.

References:

1) WALTER S. DIEHL, N. A. C. A. Report 218, 1925.
2) W. G. BROMBACHER, N. A. C. A. Report 246, 1926.

Bibliography:

For an extended bibliography on the general subject of Fluid Mechanics, the reader is referred to

National Research Council Bulletin No. 84

Report of Committee on Hydrodynamics

DRYDEN, MURNAGHAN, BATEMAN

Washington 1932.

DIVISION C

FLUID MECHANICS, PART II

By

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PREFACE

In the plan of this monograph I have followed, in general, previously trodden paths, but have deviated therefrom in some particular features which have been found desirable for a better presentation of the logical connections between the various elements of the subject.

For the purpose of a clearer and more concise presentation of the subject, the methods of vector analysis have been freely used and for the benefit of those not well acquainted with this subject, some elementary discussion of this mathematical discipline has been included.

The vector analysis employed in the present Division has certain elements of originality. The Hamiltonian rule for scalar multiplication is chosen rather than Grassmann's rule. In conjunction with this, the unit vectors effecting the computation of Cartesian coordinates are placed along the negative direction of the axes. Finally strict agreement between the association of vector multiplication and the association of vector differentiation is thus maintained. The combination of these features gives transformation equations between different vector analytical expressions without confusing minus signs, it leads to the natural sign of physical quantities, and establishes a symmetry within vector analysis not fully realized with other combinations.

The problems in three-dimensional flow have been treated with as much mathematical simplification as possible. The treatment in general follows closely that of Lamb and with similar notation in order to facilitate reference.

CHAPTER I

KINEMATICS OF FLUIDS

1. Velocity Field. The fluid, the motion of which we proceed to study, is assumed to be continuous and homogeneous, not only with respect to its physical properties but also with respect to the motion itself. We assume the differences of velocity at different points to vanish as the points approach each other. While the substantiality of the

fluid assures us of the existence of a definite velocity at each point, it is only the assumption of a continuous velocity distribution that enables us to deal numerically with the latter. We reconcile our assumption with modern atomic theory by specifying our smallest distances and regions considered separately to be large compared with atomic distances, but small when compared with the distances measured in the usual way. The mathematical *infinitely small* is an idealisation of that specification.

We conceive fluid velocities as vectors from the start, giving to the word *velocity* not only the meaning *rate of motion* but also the *direction* of this motion as well. The complexity of the velocity is therefore threefold; it requires three scalars to describe it fully. Accordingly, the addition of velocities, including their multiplication by scalars, is performed in the same manner as with the addition of forces. The components are either added or substracted separately, or, otherwise the vectors are lined up without change of individual direction, placing the front end of each one at the rear end of the next one in order, and connecting the beginning of the chain with its end.

The velocities at all points form the *vector field* of the fluid velocity. The properties of this vector field at one particular point are analyzed by bringing the velocity into relation to a small plane surface element, which at present is a mere geometric conception. Such element may likewise be conceived of as a vector, possessing magnitude of area and direction. The *direction* of such a surface element, in the vector sense, is taken as that along the normal to the surface. The one face of the surface element is supposed to be the front or positive side, and the other the rear or negative side. With surface elements of closed surfaces, we shall later consider the inside positive. Let the surface element be denoted by dS and the velocity by $\boldsymbol{V}$. We define now as the *flux* of the velocity through the surface element the volume of fluid passing through it per unit of time. This volume is a scalar, and is equal to the product of the absolute magnitude of the velocity by that of the surface element, multiplied by the cosine of the angle between the two vectors. The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is represented in Fig. 1. The rear end of the first vector $\boldsymbol{a}$ is brought into coincidence with the front end of the second vector $\boldsymbol{b}$. The angle $A\,B\,C$ is then to be considered as the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

A scalar computed from two vectors such as the flux from a velocity and a surface is called the scalar or dot product of the two vectors and is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$. According to the definition of the angle between two vectors, the dot product of a vector by itself is negative. The flux of the velocity $\boldsymbol{V}$ through the surface element $d\boldsymbol{S}$ is

$$\text{Flux} = d\boldsymbol{S} \cdot \boldsymbol{V} \tag{1.1}$$

The order of the two factors in a dot product is indifferent.

Two vectors may also be combined for the computation of a third vector, which is then called their vector or cross product, written $\boldsymbol{a} \times \boldsymbol{b}$. This vector is at right angles to the plane of the two factors, and its magnitude is equal to the product of the magnitudes of the two factors, multiplied by the sine of the angle between them. It is therefore seen to be equal in magnitude and direction to the parallelogram formed on the two vectors as sides.

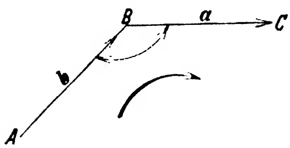

Fig. 1.

A pair of vectors as in Fig. 1 indicates a turn in the right hand or clockwise direction. In such case, with the eye above the paper looking down $\perp$ to the plane of the parallelogram, the positive direction of the vector representing the parallelogram is taken as that toward the eye and the positive face will be the nearer or upper face of the parallelogram. In view of this geometrical relation, an exchange of the factors $\boldsymbol{a}$ and $\boldsymbol{b}$ will change the sign of the cross product. The order of the two factors in a cross product is therefore of importance. The cross product of the velocity and a surface element may be called the *roll* of the velocity with respect to or over the surface

$$\text{Roll} = d\boldsymbol{S} \times \boldsymbol{V}$$

The roll indicates a passing by of fluid. As a vector it has the direction of the axis of a wheel rolling on the surface and moving with the fluid.

The transformation of vectors into ordinary algebraic quantities referred to Cartesian coordinates is accomplished by means of a set of three unit vectors, at right angles to each other and of unit length (see Fig. 2). They are generally denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, and are located relative to each other in such manner as to give relations as follows:

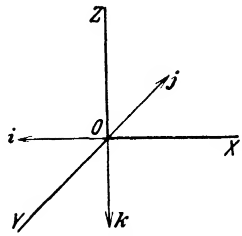

Fig. 2.

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$
$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = -1$$
$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$$
$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = 0$$

In forming such a product as $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j}$ may be considered as a rotor turning $\boldsymbol{i}$ into the position $\boldsymbol{k}$ by a 90^0 rotation in the right hand direction looking in the direction of $+Y$; or otherwise, $\boldsymbol{i}$ may be considered as a rotor turning $\boldsymbol{j}$ into the position $\boldsymbol{k}$ by a 90^0 rotation in the left hand direction looking in the direction of $+X$.

2. Surface Integrals. We proceed now to a finite surface, not plane in general, and define as the flux and roll of this surface the sum or integral of the fluxes and rolls of its elements. The rolls of the elements must be added to each other as vectors. We further assume the surface

to be closed, and specify the inside to be the positive side or face of the surface elements. The volume enclosed by the closed surface can be subdivided by one or several additional surfaces stretching across like partitions, thus forming several closed surfaces or volume elements with common partitions. However, any face common to two of these elementary closed surfaces or volume elements will be outside with respect to one and inside with respect to the other, and hence such a face will have opposite signs with respect to two such adjoining volume elements. From this it follows, that if a volume is considered as composed of several smaller elements or volumes, the flux and roll of its surface with respect to any vector field is equal to the sum of the fluxes or rolls of the surfaces of the several component parts. The summation over the inner partitions will all cancel out. We may therefore at last divide the volume into infinitely many infinitely small portions, and realize that the flux and roll of any volume is equal to the integral of the fluxes or rolls of the elements of such volume.

The flux through and roll over the surface of very small space elements do not depend on the shape of the space element but merely on its volume, and their ratios to that volume are the particular characteristics of the velocity field at that point which we shall have to use. They are called *divergence* and *rotation* respectively. The divergence is the volume of fluid leaving a small closed space element of unit volume per unit time passing through its surface from inside to outside. The rotation is twice the mean angular velocity of the fluid element[1].

The magnitude of these two quantities is obtained by differentiation. Let a point of the space be determined by Cartesian coordinates x, y, and z, and let the vector $\boldsymbol{r}$ connect the origin of the Cartesian coordinates with the point. Then using the unit vector $\boldsymbol{i}$, the expression $\boldsymbol{i} \cdot \boldsymbol{r}$ will mean the product of $\boldsymbol{r}$ as a length by unity and by the cosine of the angle between X and $\boldsymbol{r}$. But this is the component of $\boldsymbol{r}$ along X. Hence we shall have

$$x = \boldsymbol{i} \cdot \boldsymbol{r} \qquad y = \boldsymbol{j} \cdot \boldsymbol{r} \qquad z = \boldsymbol{k} \cdot \boldsymbol{r}$$

As will be seen by referring to the definition of the angle between two vectors, such as $\boldsymbol{i}$ and $\boldsymbol{r}$, this will require that $\boldsymbol{i}$ be laid off in the direction of $-x$ as previously noted, and similarly for $\boldsymbol{j}$ and $\boldsymbol{k}$. The components of the velocity $\boldsymbol{V}$ in vector form are $-u\,\boldsymbol{i}$, $-v\,\boldsymbol{j}$, and $-w\,\boldsymbol{k}$, where the components u, v, and w are scalars, *viz.* the components in the x, y, and z direction defined in the ordinary way. The choice of a small cube with the sides dx, dy, and dz as space element for the determination of the divergence and of the rotation gives then the expressions:

[1] See Division A VI 8.

$$\left.\begin{aligned} \textit{Divergence} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ &= \boldsymbol{i}\cdot\frac{\partial \boldsymbol{V}}{\partial x} + \boldsymbol{j}\cdot\frac{\partial \boldsymbol{V}}{\partial y} + \boldsymbol{k}\cdot\frac{\partial \boldsymbol{V}}{\partial z} \end{aligned}\right\} \tag{2.1}$$

$$\textit{Rotation} = \boldsymbol{i}\times\frac{\partial \boldsymbol{V}}{\partial x} + \boldsymbol{j}\times\frac{\partial \boldsymbol{V}}{\partial y} + \boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z} \tag{2.2}$$

The x component of the rotation for instance, is $\partial w/\partial y - \partial v/\partial z$.

A shorter way of expressing the rule for the computation of the divergence and the rotation, and one, moreover, which makes the introduction of Cartesian coordinates unnecessary, is the introduction of a symbolic vector, called "*del*" and written ∇. This symbol expresses the differentiation above. It stands for the operation

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{2.3}$$

We may then write the vector $\boldsymbol{V}$ and the operator ∇ as follows:

$$\boldsymbol{V} = -\boldsymbol{i}\,u - \boldsymbol{j}\,v - \boldsymbol{k}\,w$$

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

If now we go through the detail of applying each of the members on the lower line to each of those on the upper line we shall have nine partial results. These will fall into the two groups as follows:

$$\nabla\cdot\boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla\times\boldsymbol{V} = \boldsymbol{i}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \boldsymbol{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \boldsymbol{k}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

and we see at once

$$\textit{Divergence} = \nabla\cdot\boldsymbol{V} \tag{2.4}$$

$$\textit{Rotation} = \nabla\times\boldsymbol{V} \tag{2.5}$$

∇ may be used as a vector, with certain exceptions also found with the use of the ordinary differentiation symbol "d" in calculus. These symbols are generally used like factors. However, when connected with a product of several factors, like $d\,(a\,b\,c)$ they have to be multiplied by each of the factors separately and the products added to each other, thus:

$$d\,(a\,b\,c) = (d\,a)\,b\,c + a\,(d\,b)\,c + a\,b\,(d\,c)$$

In calculus, the order is immaterial, but not so in vector analysis. Likewise, if the product containing d or ∇ is multiplied by further factors, such as $(d\,a)\,c$, the association between d and a must not be changed. In a word, ∇ is a differentiation symbol, treated otherwise like a vector.

We can now express our former result by saying that the scalar surface integral of a velocity field is equal to the space integral of the divergence of that velocity field. Denoting the element of area by $d\boldsymbol{S}$ and that of volume by dQ, this theorem takes the form

$$\int d\boldsymbol{S}\cdot\boldsymbol{V} = \int dQ\,(\nabla\cdot\boldsymbol{V}) \tag{2.6}$$

This is known as Gauss' theorem and is the most important one of many analogous theorems. The other result, for the vector surface integral takes the form

$$\int d\boldsymbol{S} \times \boldsymbol{V} = \int dQ\,(\nabla \times \boldsymbol{V}) \tag{2.7}$$

This has not received any special name.

3. Line Integral. In most treatises on the subject, the rotation is derived by means of the scalar line integral $\int d\boldsymbol{s} \cdot \boldsymbol{V}$ taken along a curve of which $d\boldsymbol{s}$ is an element (vector sense). We see again, that if the line is closed, and any surface enclosed by it is divided into smaller portions by additional lines connecting points of the original line, the same arguments hold as with the space integral. The line integral, taken clockwise as seen from one side of the surface around its boundary is equal to the sum of the line integrals taken around its portions, because the contributions along the dividing lines cancel out, each element being traversed in two opposite directions. The ratio of the line integral along a closed curve to the small area of the surface enclosed, approaches again a limit independent of the shape of the curve. This is the magnitude of the component of the rotation at right angles to the surface[1].

The theorem that the scalar line integral of a vector field is equal to the scalar surface integral of the rotation of the same vector field over the enclosed surface is known as Stokes' theorem[2], and is expressed as follows

$$\int d\boldsymbol{s} \cdot \boldsymbol{V} = \int d\boldsymbol{S} \cdot \nabla \times \boldsymbol{V} \tag{3.1}$$

The scalar line integral receives a special name in the important but special case where the velocity field is of such description that the scalar line integral along a closed line is always zero. Stokes' theorem (3.1) shows that this takes place if the rotation is zero at all points. In that case, the scalar line integral between two points does not depend on the shape of the line connecting the two points along which the integration has been taken, since the difference of the two integrals (as along 1 A 2 and 1 B 2 Fig. 3) would be the line integral taken around the closed curve formed by these two paths, and hence zero under the present assumption. Hence, choosing one fixed point as origin, the scalar line integral depends solely on the location of the second point at the end of the integration path; and hence the scalar line integral is then an ordinary function of the space coordinates—in other words, a scalar field. It is then called the scalar potential of the velocity field, or merely velocity potential or potential. Conversely, the vector field $\boldsymbol{V}$

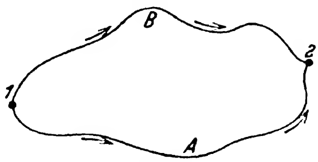

Fig. 3.

[1] See Division A VI 9.
[2] See Division A IX 3.

belonging to a scalar potential field we shall call the *antigradient*. It can be computed from the potential (denoted by φ) by differentiation. Written as the sum of vector components, the antigradient is

$$\boldsymbol{V} = \boldsymbol{i}\frac{\partial \varphi}{\partial x} + \boldsymbol{j}\frac{\partial \varphi}{\partial y} + \boldsymbol{k}\frac{\partial \varphi}{\partial z}$$

This can be written $\nabla \varphi$ and since $d\boldsymbol{s} = -\boldsymbol{i}\, dx - \boldsymbol{j}\, dy - \boldsymbol{k}\, dz$, it follows that $d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = d\boldsymbol{s} \cdot \nabla \varphi$

The change of φ at the two ends of the line element $d\boldsymbol{s}$ is equal to the dot product of the line element and the antigradient.

The quantity $\boldsymbol{V}$, defined as above, we shall call the *derivation* (of φ in this case). A derivation is generally obtained by differentiation, it is generally a derivative, but in order to be a derivation, it must further stand in the described relation to the field from which it is derived.

The magnitude of the component of the antigradient in any direction is equal to the rate of change of φ per unit length in that direction.

The antigradient is sometimes called the vectorial rate of change of its scalar field. Such expression requires a definition of vector division, which we wish to avoid.

We thus have, for any scalar function

$$\begin{aligned} \text{Antigradient} &= \nabla \varphi \\ \text{Gradient} &= -\nabla \varphi \end{aligned}$$

The symbol ∇ has thus occurred in three forms: multiplied by a scalar without multiplication symbol, the *antigradient*; its dot product by a vector, the *divergence*; and its cross product by a vector, the *rotation*. This exhausts the possibilities, as far as we have discussed vector combinations. All three combinations represent important properties of the velocity distribution about a point in a velocity field. The antigradient is the derivation of a scalar field, the divergence and rotation are merely derivatives of a vector field, but not its derivation.

4. Scalar Triple Vector Products. The product of two vectors can again be multiplied by a third vector, as on the right hand side of Stokes' equation (3.1). Such combination of three vectors is called a triple product. The two vectors first multiplied by each other are said to be directly associated.

The triple product may be a scalar, *viz.* the scalar product of a cross product by a vector. Its magnitude is equal to the volume generated by the parallelogram formed by the two vectors directly associated, moved along the third vector. Its magnitude is therefore equal to the volume of the parallelepiped with the three vectors as edges. It follows that the dot and cross in a scalar triple product may be exchanged, which involves the change of the association

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} \tag{4.1}$$

The order of the factors may also be changed, and a short study will show that the sign changes with a change of the cyclic order. (*Cyclic rule.*) Each scalar triple product can therefore be written in twelve different ways, the transformations are extremely simple and the only rule to be observed is the cyclic rule for the sign.

5. Vectorial Triple Vector Products. We proceed to the vectorial triple product, which may either be the product of a vector by the dot product of two other vectors, called the dot triple product, or the cross product of a cross product by a third vector, called a cross triple product, thus: $\boldsymbol{a} \cdot \boldsymbol{b}\,\boldsymbol{c}$ or $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

Either vector triple product can be written in four different ways without change of association, thus:

$$\left.\begin{aligned} &\boldsymbol{a} \cdot \boldsymbol{b}\,\boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{a}\,\boldsymbol{c} = \boldsymbol{c}\,\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c}\,\boldsymbol{b} \cdot \boldsymbol{a} \\ &(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -(\boldsymbol{b} \times \boldsymbol{a}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{a}) \end{aligned}\right\} \quad (5.1)$$

From the basic rules for multiplication it follows that the dot product does not change its sign when thus transformed. The cross product does change sign with a single change in the cyclic order. With two changes, the sign remains the same. This may be briefly expressed by saying that the product changes sign with a change of the center vector. This is known as the *centric* rule. Each vector triple product can further be written as sum of two other products containing the same three factors but with changed association. We thus have:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b}\,\boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{c}\,\boldsymbol{b} \quad (5.2)$$

This relation may be proven as follows:

Put
$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{i}\,a_1 + \boldsymbol{j}\,a_2 + \boldsymbol{k}\,a_3 \\ \boldsymbol{b} &= \boldsymbol{i}\,b_1 + \boldsymbol{j}\,b_2 + \boldsymbol{k}\,b_3 \\ \boldsymbol{c} &= \boldsymbol{i}\,c_1 + \boldsymbol{j}\,c_2 + \boldsymbol{k}\,c_3 \end{aligned}$$

If we then form in detail the products indicated in (5.2) we shall find that the equation is verified. For orthogonal vectors, the result is readily proven by substituting for $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, all combinations of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. The proposition may also be proven geometrically.

6. Helmholtz' First Theorem. We now proceed to study triple products containing two identical vectors, these two being moreover the symbolic vector ∇. The divergence of the rotation of a vector field would have to be written: $\nabla \cdot \nabla \times \boldsymbol{V} = 0$ (6.1)

This is a scalar triple product, and is identically zero, because the volume of a parallelepiped with two coinciding sides is zero, or because it can be transformed into $\nabla \times \nabla \cdot \boldsymbol{V}$ and the cross product of a vector by itself is zero. We realize therefore that the rotation of a vector field has always the divergence zero at all points. This constitutes Helmholtz' first theorem regarding fluid motion, a purely geometrical theorem.

A vector field not free from divergence cannot be a rotation field, there cannot exist a vector potential of which it is the rotation.

We have deduced already from Stokes' theorem (3.1) that the rotation of a gradient or of an antigradient is zero. This can likewise easily be committed to memory by writing the rotation in vector symbols. It takes the form

$$\nabla \times \nabla \varphi = 0 \tag{6.2}$$

But $\nabla \times \nabla$ is the symbolic cross product of two parallel vectors and is thus necessarily zero.

7. Stream-lines. Before going further in the analysis of velocity distribution at a point, we may use the results so far obtained for drawing a mental picture of the velocity distribution as a whole. We define as stream-lines, lines parallel everywhere to the velocity. In general the velocity at every point changes its direction continually with time, and then the stream-lines are by no means identical with the paths of the fluid particles. In the special case of "steady" motion, where all velocities remain permanently the same at all points, the stream-lines coincide with the path lines of the fluid particles.

If the fluid is incompressible, each particle retains its volume, and since no fluid is created nor annihilated at any point, the flux into every portion of the space must be equal to the flux out of it, and hence the total flux as defined in 1 must be zero. We recognize therefore that the flow of an incompressible fluid is free from divergence. This condition is often called the condition of continuity.

Such continuity being assumed, the picture of the stream-lines can be developed further. Consider all stream-lines passing through the points of a small closed curve. They cover the surface of a tube, through which the fluid is flowing. Since the fluid is at present assumed to be incompressible, an equal volume of fluid flows through each cross section of the stream-tube. The entire space can thus be divided into such stream-tubes, and in particular the division can be made so that there flows through each cross section of each stream-tube one unit volume per unit time. If desirable, the unit may be chosen very small. It is then sufficient to represent each unit stream-tube by its axis, which we call a unit stream-line or shorter, a stream-line.

From the mathematical point of view, these stream-lines have much in common with the lines of force of magnetic and of other force fields. It is at once seen that the number of stream-lines passing through a surface gives the flux of the latter. The number passing through a unit area normal to the lines gives the velocity, so that the velocity is largest where the stream-lines crowd the closest.

Since the rotation of the velocity is always free from divergence, the same picture can be used for the rotation vectors. Unit rotation lines are called vortex lines. According to Stokes' theorem, the number

of vortices passing through a surface is equal to the scalar line integral around its circumference. The density of the vortex lines is an indication of the magnitude of the angular velocity of the fluid particles at each point.

The picture of the motion of an incompressible fluid consisting of a system of stream-lines, and another system of vortex lines is the best one devised. There are cases, imaginable at least as an idealization, where vortex lines crowd together to an infinite density. This takes place if two portions of fluid glide along each other with a finite step in the velocity component tangential to the surface separating them. The vortex lines are then concentrated in that surface, and their number or density has to be computed by means of Stokes' theorem. The infinitely large angular velocity within a layer of infinitely small thickness is the mathematical idealization of a very large angular velocity in a very thin layer, like the oil film in a bearing, or, in a way, like the balls in a ball bearing.

8. Dyadic Multiplication. The discussion of the vectorial triple products has shown that it is not permitted to change the association of the factors. This impossibility constitutes a stumbling block to an intelligent manipulation of vectorial triple products. The sum of several of them with an associated factor in common cannot be written as the product of this factor by an expression containing the remaining factors. Furthermore, if the differential symbol ∇ enters as an associate factor, the differentiation will extend to the other associate factor only, and we have no means of indicating such combination with the differentiation extending to the factor non-associated. In equation form, we cannot yet write $\boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{d}\,\boldsymbol{e}=\boldsymbol{a}$ *times a quantity;*
and in $\boldsymbol{a}\cdot\nabla\boldsymbol{c}$ the differentiation extends to $\boldsymbol{a}$ and we have not as yet any way of writing this so that the differentiation extends to $\boldsymbol{c}$.

In order to remove these difficulties, and to provide for the necessary expressions, we introduce the dyadic multiplication symbol ";" defined by

$$\left.\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}&=\boldsymbol{a}\cdot\boldsymbol{b}\,;\boldsymbol{c}=\boldsymbol{c}\,;\boldsymbol{b}\cdot\boldsymbol{a}\\ \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{d}\,\boldsymbol{e}+\ldots&=\boldsymbol{a}\cdot(\boldsymbol{b}\,;\boldsymbol{c}+\boldsymbol{d}\,;\boldsymbol{e}+\ldots)\end{aligned}\right| \tag{8.1}$$

There is no difference between the value of $\boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}$ and $\boldsymbol{a}\cdot\boldsymbol{b}\,;\boldsymbol{c}$, but in the dyadic form $\boldsymbol{b}$ and $\boldsymbol{c}$ are considered associate; in the dot triple products, $\boldsymbol{a}$ and $\boldsymbol{b}$ are. Accordingly, in $\boldsymbol{a}\cdot\nabla\boldsymbol{c}$ the differentiation extends to $\boldsymbol{a}$, but in $\boldsymbol{a}\cdot\nabla\,;\boldsymbol{c}$ the differentiation extends to $\boldsymbol{c}$.

From $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}-\boldsymbol{a}\cdot\boldsymbol{c}\,\boldsymbol{b}$, see (5.2)

follows immediately:

$$\left.\begin{aligned} &\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}\,;\boldsymbol{c}-\boldsymbol{a}\cdot\boldsymbol{c}\,;\boldsymbol{b}\\ \text{or}\quad&\boldsymbol{a}\cdot\boldsymbol{b}\,;\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{a}\cdot\boldsymbol{c}\,;\boldsymbol{b}\end{aligned}\right\} \tag{8.2}$$

This holds also for the sum of any number of dyadics,

$$\left.\begin{aligned} \boldsymbol{a}\cdot(\boldsymbol{b}\,;\boldsymbol{c}+\boldsymbol{d}\,;\boldsymbol{e}+\ldots)&=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{d}\times\boldsymbol{e}+\ldots)+\\ &+\boldsymbol{a}\cdot(\boldsymbol{c}\,;\boldsymbol{b}+\boldsymbol{e}\,;\boldsymbol{d}+\ldots)\end{aligned}\right| \tag{8.3}$$

9. The Derivation of a Vector Field. As the derivation of a vector field $\boldsymbol{V}$ we define a quantity the product of which by a line element $d\boldsymbol{s}$ gives the change of $\boldsymbol{V}$ between the beginning and end of the line element. We anticipate that the derivation will be a derivative of the vector field, that is, it will be computed from it by differentiation. The rotation of $\boldsymbol{V}, \nabla \times \boldsymbol{V}$ and the divergence of $\boldsymbol{V}, \nabla \cdot \boldsymbol{V}$, which are likewise derivatives of $\boldsymbol{V}$, do not, however, play the part of the derivation of $\boldsymbol{V}$ in general.

It is easily seen, that the dyadic product of ∇ and $\boldsymbol{V}$

$$\nabla ; \boldsymbol{V} = \boldsymbol{i} ; \frac{\partial \boldsymbol{V}}{\partial x} + \boldsymbol{j} ; \frac{\partial \boldsymbol{V}}{\partial y} + \boldsymbol{k} ; \frac{\partial \boldsymbol{V}}{\partial z}$$

is quite generally the derivation of $\boldsymbol{V}$. Multiplying $\nabla ; \boldsymbol{V}$ by

$$d\boldsymbol{s} = -\boldsymbol{i}\,dx - \boldsymbol{j}\,dy - \boldsymbol{k}\,dz$$

gives
$$d\boldsymbol{s} \cdot \nabla ; \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial x} dx + \frac{\partial \boldsymbol{V}}{\partial y} dy + \frac{\partial \boldsymbol{V}}{\partial z} dz = d\boldsymbol{V}$$

which equation represents exactly the relation by which the derivation is defined.

In order to obtain the change of $\boldsymbol{V}$, the derivation dyadic has to be multiplied by $d\boldsymbol{s}$ so that $d\boldsymbol{s}$ and ∇ become adjacent to each other.

10. Acceleration of Fluid Particles. We are now prepared to write the equation for the acceleration of the fluid particles, which will be needed in the next chapter for the application of Newton's principles to fluid motion. This acceleration is the rate of change of the velocity of an individual fluid particle, and must not be confused with the rate of change of the velocity at one particular point of space, which point is occupied by different particles one after the other. The latter rate of change is sometimes called the local rate of change, but would better be called the time derivation either of the velocity or of any other property of the fluid particles under discussion. The former rate of change, referring to the particle itself, is usually called the absolute rate of change. The difference of the two is called the progressive rate of change.

The absolute rate of change of any property $\boldsymbol{b}$ is accordingly composed of two parts; of the time derivation of that property conceived as a function of the space, and of the progressive rate of change arising from the fact that the fluid particle is travelling between regions of different values of $\boldsymbol{b}$. If the fluid motion is steady, the change arises from such travel only, and is then equal to the progressive rate of change. If the motion is unsteady, there is further a finite contribution of the time derivation.

Taking the former of these first, the particle travels, during the small time interval dt through a distance $d\boldsymbol{s} = \boldsymbol{V} dt$.

The difference of $\boldsymbol{b}$ at the two ends of the path $d\boldsymbol{s}$, as we have seen, is equal to the product of $d\boldsymbol{s}$ by the derivation dyadic of $\boldsymbol{b}$. Hence,

the rate of change per unit time is equal to the same product divided by dt, and hence is equal to the product of $d\boldsymbol{s}/dt$, or $\boldsymbol{V}$, by the derivation dyadic. Thus
$$\frac{D\boldsymbol{b}}{Dt} = \boldsymbol{V}\cdot(\nabla;\boldsymbol{b})$$

If the flow is unsteady, we have, in addition to this term, the time derivation; for this rate affects the entire small region considered. The absolute rate of change of a vector property $\boldsymbol{b}$ of the fluid can therefore be expressed as follows, denoting the absolute rate of change by a capital D and the time derivation by a round ∂:

$$\frac{D\boldsymbol{b}}{Dt} = \frac{\partial\boldsymbol{b}}{\partial t} + \boldsymbol{V}\cdot(\nabla;\boldsymbol{b}) \tag{10.1}$$

If the vector property $\boldsymbol{b}$ is identical with the velocity $\boldsymbol{V}$ itself, the absolute rate of change becomes identical with the acceleration, and we obtain:
$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial\boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot(\nabla;\boldsymbol{V}) \tag{10.2}$$

The dyadic expression in (10.2) can at once be converted into one containing ordinary triple products only. Such conversion is always possible, and treatises on vector analysis contain transformation equations serving for all such cases.

In the present case, the transformation can easily be deduced separately as follows.

By (5.2) $\qquad \boldsymbol{V}\cdot\nabla;\boldsymbol{b} = \boldsymbol{V}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{V}\cdot(\boldsymbol{b};\nabla)$

Now we have $\qquad \nabla\boldsymbol{V}\cdot\boldsymbol{b} = \nabla;\boldsymbol{V}\cdot\boldsymbol{b} + \nabla;\boldsymbol{b}\cdot\boldsymbol{V}$

The product of ∇ by a product of two factors is expressed as the sum of two products formally equal to it, ∇ being associated to one factor only in the two right hand expressions. Now put $\boldsymbol{b} = \boldsymbol{V}$

$$\nabla\boldsymbol{V}\cdot\boldsymbol{V} = 2\nabla;\boldsymbol{V}\cdot\boldsymbol{V} \quad \text{or} \quad \boldsymbol{V}\cdot\boldsymbol{V};\nabla = \frac{1}{2}\nabla\boldsymbol{V}\cdot\boldsymbol{V}$$

Inserting this gives at last

$$\boldsymbol{V}\cdot\nabla;\boldsymbol{V} = \frac{1}{2}\nabla(\boldsymbol{V}\cdot\boldsymbol{V}) + \boldsymbol{V}\times(\nabla\times\boldsymbol{V})$$

This gives for the acceleration

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial\boldsymbol{V}}{\partial t} + \frac{1}{2}\nabla(\boldsymbol{V}\cdot\boldsymbol{V}) + \boldsymbol{V}\times(\nabla\times\boldsymbol{V}) \tag{10.3}$$

or, if U denotes the absolute magnitude of the velocity

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial\boldsymbol{V}}{\partial t} - \frac{1}{2}\nabla U^2 + \boldsymbol{V}\times(\nabla\times\boldsymbol{V}) \tag{10.4}$$

This is the desired expression for the fluid acceleration, written without the use of the dyadic multiplication symbol.

A second equation, which we shall need in the next chapter, refers to the rotation, rather than to the velocity. We shall not, however,

compute the absolute rate of change of the rotation, but the rate of change of the number of vortex lines passing through a closed line moving with the fluid.

Since the divergence of the rotation is zero, (6.1), no vortex line can begin or end inside the fluid, and hence no surface connecting the closed line can increase or decrease the number of intersecting vortex lines by passing such beginnings or ends. The rate of change is thus restricted to the number of vortex lines intersecting with the closed line as it travels with the fluid particles. During the time element dt this line travels through a ring shaped surface with surface area elements $d\boldsymbol{s} \times \boldsymbol{V} dt$, where $d\boldsymbol{s}$ is the element of the closed line. The flux of the vortex lines through all such elements [see (4.1)], is

$$\int d\boldsymbol{s} \times \boldsymbol{V} \cdot (\nabla \times \boldsymbol{V})\, dt = -\int (\nabla \times \boldsymbol{V}) \times \boldsymbol{V} \cdot d\boldsymbol{s}\, dt \tag{10.5}$$

The scalar line integral thus obtained is now transformed into a surface integral by the use of Stokes' rule (3.1). This gives the rate of change of the vortex flux in a steady flow in the form

$$-\int d\boldsymbol{S} \cdot \nabla \times [(\nabla \times \boldsymbol{V}) \times \boldsymbol{V}] \tag{10.6}$$

Since $\nabla \times [(1/2)\, \nabla \boldsymbol{V} \cdot \boldsymbol{V}] = 0$, this being the rotation of a gradient, we can add it to the expression in the bracket (10.6), and we further add the rate of change arising from the lack of steadiness of the flow, and obtain thus for the entire rate of change

$$\int d\boldsymbol{S} \cdot \nabla \times \left[\frac{\partial \boldsymbol{V}}{\partial t} + \frac{1}{2} \nabla \boldsymbol{V} \cdot \boldsymbol{V} + \boldsymbol{V} \times (\nabla \times \boldsymbol{V})\right]$$

This is equal to
$$\int d\boldsymbol{S} \cdot \nabla \times \frac{D\boldsymbol{V}}{Dt} \tag{10.7}$$

as seen by comparison with (10.3).

It results therefore that the rate of change of the number of vortex lines within a closed circuit moving with the fluid is equal to the scalar surface integral of the rotation of the acceleration of the fluid particles. It is zero, in particular, if the rotation of the acceleration is zero.

11. Boundary Conditions. Superposition. We close this brief abstract of fluid kinematics by two explanations for later reference. A velocity distribution $\boldsymbol{V}_3$ can be computed from two other distributions $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ by means of the equation

$$\boldsymbol{V}_3 = \boldsymbol{V}_1 + \boldsymbol{V}_2$$

This is called *superposition*; the third velocity distribution is said to be the superposition of the two others or rather the result of such superposition, and these are said to be superposed the one on the other. It follows from the definition of divergence and rotation, that the divergence and the rotation of the velocity field obtained from the superposition is equal to the sum of the divergence and rotation of the two (or more) superposed fields. If all of them are free from divergence or rotation, the superposed field is likewise.

We have finally to discuss the conditions with which the velocity distribution must comply at the points of a solid wall in contact with the fluid, such as the surface of a solid moving through the fluid. No surface element restricts directly the velocity components parallel to the wall, that is its roll. It prevents however the fluid from passing through it. Hence, if the solid surface element is not moving, the flux through it must be zero. If it is moving with the velocity u, its flux with respect to this velocity must be equal to the flux with respect to the velocity $\boldsymbol{V}$ of the fluid, in order to make the absolute flux relative to the surface equal to zero; or otherwise expressed, the normal component of the wall velocity must be equal to the normal component of the fluid velocity. The tangential component of the fluid velocity is not directly restricted, at least so long as the fluid is non-viscous.

CHAPTER II

DYNAMICS OF FLUIDS

1. Pressure. The fluid particles move in compliance with Newt[illegible] principles, under the influence of externa forces, such as gravity, [illegible] under the action of adjacent particles.

We assume all actions between adjace normal to their common boundary throug place. This assumption of non-viscosity red the fluid to a pure pressure. The resultant adjacent particles on the fluid particle consid the buoyancy of the particle caused by the contemplation of a small rectangular space dz shows this buoyancy to be equal to the the particle multiplied by the antigradient of the pressure, and $\boldsymbol{F}$ the force per unit volu

$$\boldsymbol{F} = \nabla p$$

The external force per unit volume may forces $\boldsymbol{F}$ and $\boldsymbol{F}_1$ is proportional to the a Hence I (10.4) gives

$$\varrho \left[\frac{\partial \boldsymbol{V}}{\partial t} - \frac{1}{2} \nabla U^2 + \boldsymbol{V} \times (\nabla \right.$$

where ϱ denotes the density.

2. Materiality of the Vortices. We change of the vortex lines moving wit of the acceleration. This rotation rotation of the force (1.1) divided

$$\nabla \times ($$

In the case of constant density, rotation of the gravity field is ze

of the pressure is likewise zero. [See I (6.2).] If the density is not constant but the fluid elastic, the density can at least be assumed to be a function of the pressure only. If further, in this latter case of elastic fluids, the effect of gravity be neglected, the rotation of the acceleration becomes $\nabla \times \frac{DV}{Dt} = \nabla \times \left(\nabla p \, \frac{1}{\varrho}\right)$ (2.1)

But $\varrho = f(p)$ and $p/\varrho = p/f(p) = f_1(p)$. Then $\nabla \times [\nabla f_1(p)] = 0$. [See I (6.2).]

We have thus arrived at Helmholtz' second vortex theorem. In the absence of external forces, the vortex lines move with the fluid particles; this holds also if the external forces are distributed irrotationally, like the gravity forces of homogeneous incompressible fluids.

This theorem forms the basis of the theory of the motion of perfect fluids, because it permits a solution of (1.1) without the computation of the pressure. If all velocities are given at the beginning of the investigated interval, there are still two functions undetermined in (1.1), pressure and the local rate of change of the velocity field. Helmholtz' nd theorem provides for the computation of the local rate of change,
it ... li ... the vortex field after the next time interval has
...e then moved as indicated by the velocity
Helmholtz' second law gives a combination
of change, which when integrated according
dependent of the path of integration and is
n. Otherwise expressed, Helmholtz' theorem
em to a kinematic problem.

s a special case of Helmholtz' theorem, it
continues to move without vorticity, if it
f external forces are absent. This appears
imagined to be composed of small perfectly
tely smooth sphere cannot be turned or
e forces. In most aeronautical problems,
ave been at rest initially, and to have been
nent of one or several solids. The motion
and hence, in consequence of Helmholtz'
is important to realize that this absence
ssumption made for the simplification
lirectly from the assumption of absence
the fluid.

ational flows always possess a velo-
ich is the velocity. Since a scalar
there being only one dependent
tion of such a flow is generally
computed, the velocity follows,

We express therefore the condition of incompressibility of the fluid, the so-called condition of continuity, in terms of the potential. This is a kinematic condition and puts a restriction on the potential, having the form of a partial differential equation. We obtain it by putting the velocity equal to the antigradient of the potential; $V = \nabla \varphi$. The condition of continuity is that of absence of divergence $(\nabla \cdot \nabla \varphi)$ and hence the divergence of the antigradient of the potential is seen to be zero

$$\nabla \cdot \nabla \varphi = 0$$

or; in ordinary Cartesian coordinates

$$\frac{d^2 \varphi}{d x^2} + \frac{d^2 \varphi}{d y^2} + \frac{d^2 \varphi}{d z^2} = 0 \tag{3.1}$$

This is Laplace's equation of continuity for incompressible fluids in Cartesian coordinates. The time t does not occur in it. This does not mean that (3.1) holds only for steady flows, it holds equally for steady and unsteady flows. It means that there is no fundamental difference between the momentary velocity distribution of a steady flow and of an unsteady flow, but every such velocity distribution may continue as a steady flow, or change according to the sequence of the boundary conditions.

4. Bernoulli's Pressure Equation. If the motion is irrotational, (1.1), in the absence of external forces, reduces to

$$\varrho \left(\frac{\partial}{\partial t} \nabla \varphi - \frac{1}{2} \nabla U^2 \right) = \nabla p$$

This can be integrated,

$$\varrho \frac{\partial \varphi}{\partial t} - \frac{\varrho U^2}{2} = p + \text{const.} \tag{4.1}$$

This is Bernouilli's pressure equation. If in particular the flow is steady, the equation reduces farther to

$$p = -\frac{\varrho U^2}{2} + \text{const.} \tag{4.2}$$

This shows that the pressure in a steady irrotational flow depends on the velocity only. It has its maximum where the velocity is smallest or zero. This will appear plausible if we consider that the fluid is accelerated when flowing from a region of higher pressure to one of lower pressure.

It can easily be proven in an elementary way that (4.2) holds along any stream-line of a steady flow of an ideal incompressible fluid, whether this flow is irrotational or not. The significance of (4.2) lies in the fact that the constant has the same value for all stream-lines.

It has become usual to denote the term $(1/2)\, \varrho\, U^2$ in Bernoulli's equation as "dynamic pressure". Hence it is necessary to introduce a special name for the actual pressure in the fluid. It is called "static pressure". Bernoulli's equation states that in a steady flow the sum

of the static and the dynamic pressure is constant throughout the fluid, not only along each stream-line. This sum is sometimes called "total pressure".

5. Fictitious Flows. In the investigation of fluid forces experienced by moving solids surrounded by fluid, it is often convenient to assume the fluid motion to extend inside of the solid, that is to replace the effect of the solid surface by the effect of a fictitious motion of the fluid. Such fictitious fluid motions differ from actual ones in that the motion is supposed to have divergence in certain regions or at certain points in spite of supposed constant density. This can only be, if some fluid is supposed to be created or annihilated. The fluid newly created or annihilated has to be assumed to possess at first and at last equal velocity at all such points, as otherwise the mechanical system would be incomplete, and the usefulness of the fiction impaired. We assume this velocity to be zero. The mental picture of the complete flow picture consists then of infinitely narrow tubes, too narrow to disturb the flow, into which tubes the fluid vanishes at certain points called sinks, and appears again at certain other points, called sources. There may further be regions of vorticity in these fictitious flows.

The force equation, (1.1) expresses the magnitude of the external forces necessary to enforce the region with vorticity to remain steady, for in the absence of such external forces, the vortex lines move with the fluid and in general the flow is then not steady. The term in (1.1) denoting this part of the force is

$$\varrho \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) \tag{5.1}$$

The fictitious divergence requires further, per unit volume, a force for accelerating the fluid delivered by the sources from the velocity zero to the velocity $\boldsymbol{V}$. The volume delivered per unit time is $\nabla \cdot \boldsymbol{V}$ and its mass is $\varrho \nabla \cdot \boldsymbol{V}$, hence this force is

$$\varrho \boldsymbol{V} \nabla \cdot \boldsymbol{V} \tag{5.2}$$

These two classes of external forces in a steady fictitious flow can be more practically described by stating that every unit vortex line experiences a force at right angles to it and to the velocity at that point, proportional to the length of the vortex element, to the density of the fluid and to the velocity. Each unit source, that is one with unit flux, experiences a force parallel to the velocity and proportional to it, and to the density. The vortex force is directed towards the direction of smaller velocity, the corresponding opposite reaction of the fluid, in compliance with Bernoulli's pressure equation. The external source force is in the direction of the velocity, the sink force opposite. Equation (1.1) completed by the term for fictitious divergence, reads

$$\boldsymbol{F}_1 + \nabla p = \varrho \left[\frac{\partial \boldsymbol{V}}{\partial t} - \frac{1}{2} \nabla V \cdot V + \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) + \boldsymbol{V} \nabla \cdot \boldsymbol{V} \right] \tag{5.3}$$

6. Physical Interpretation of the Velocity Potential. Such interpretation can be obtained from (4.1) and is useful for understanding and memorizing many important applications.

The part of the pressure arising from the unsteadiness of the flow is seen in (4.1) to be proportional to the time rate of change of the potential. Suppose now this rate of change of the potential to become larger and larger. The term $\partial \varphi / \partial t$ then becomes large when compared with the term $\frac{1}{2} U^2$ and this latter term can be neglected when compared with the former one, so that the equation reduces to

$$p = \varrho \frac{\partial \varphi}{\partial t} \tag{6.1}$$

The flow may now be created from rest during a very short time at a very large and constant rate of change. Equation (4.1) can then be integrated with respect to time and gives

$$\int_0^t p\, d t = \varrho \varphi \tag{6.2}$$

The velocity potential, multiplied by the density, is therefore seen to be the impulsive pressure necessary to create the flow from rest. Such impulsive pressure, very large and acting a short time only, like the blow of a hammer, is measured by the product of pressure and time.

CHAPTER III

MOTION OF SOLIDS IN A FLUID

1. The Velocity Distribution. We assume the fluid to fill the entire space outside a solid immersed in it. The solid moves with constant velocity and direction, causing thereby the fluid to move with a definite velocity distribution. This is not a steady motion as defined in I 7. The configuration of velocity remains unchanged relative to the solid, but it moves along with it, whereas the condition of steadiness requires all velocities to remain constant at fixed points of the space. The same flow may however be described also as a steady motion, *viz.* the motion of the fluid as it appears to an observer moving with the solid. This steady flow is different from the first one with vanishing velocity at large distance, but can easily be obtained or computed from it by superposing the reversed constant velocity of the solid.

Bernoulli's equation II (4.1), becomes particularly simple for steady flows, and since the computation of the pressure is often the principal aim of the investigation, it is often found convenient to use the steady motion as seen by an observer moving with the solid. For other purposes one or the other of the two flows may be preferable, and both are found in use.

While it is *a priori* clear that under the described conditions the fluid can have one velocity distribution only, it is nevertheless important to investigate the question as to whether the mathematical conditions so far developed actually determine one distribution only, or more than one, and in the case that this depends on the circumstances of the flow, to fix clearly the conditions for the singularity of the solution. The question can be easily settled.

The solution obtained must meet the conditions of freedom from divergence and from rotation. Suppose two different velocity distributions to answer this condition, and to further comply with the boundary condition at infinity and at the points of the surface of the solid. Picture now the superposed flow obtained from subtracting one solution from the other. This third flow has no normal velocity components at points of the surface and it has the velocity zero at infinity. Unless it vanishes at all points, it must be possible to draw the stream-lines. These stream-lines cannot end at the solid surface, for want of a normal velocity component, nor at infinity where the velocity is zero. Hence there remains only the possibility of closed curves. Suppose now there to be no tunnel-like openings in the solid, like in a ring. It is then possible to connect any possible closed stream-line by a surface lying entirely in the fluid. The scalar line integral along the closed stream-line is not zero, because the velocity does not change its direction relative to the direction of integration. Hence from Stokes' theorem this integral indicates a finite number of vortex lines within the closed stream-line, and this is in contradiction to the assumed absence of rotation. Hence the two velocity distributions are identical.

It appears however that the singleness of the solution can only be expected for a singly connected space, and in what follows, we shall always assume a space meeting this condition.

2. Apparent Mass. We continue the study of the fluid motion caused by the solid moving uniformly, and compare the different velocity distributions corresponding to parallel motions of the solid at different velocities. They can be obtained by the superposition of the same flow upon itself. It can therefore be seen that the shape of all the stream-lines is the same for these different velocities, the magnitudes of the fluid velocities at each point being proportional to the velocity of the body. Any one such velocity distribution can be computed from the others by multiplying all velocities, or the velocity potential, by the ratio of the velocities of the solid. This corresponds exactly to the change of the conditions at the boundary of the solid, as the normal velocity components of the solid are of course proportional to its velocity, thus leaving the rotation and divergence zero, if they were so at the start.

Hence, although the fluid particles move in a very complicated way, the motion of each particle takes place as though caused by a mechanical connection between it and the solid, enforcing a motion of the fluid at that point in a certain direction and geared down or up in a certain ratio.

This leads to an important conclusion regarding the kinetic energy of the fluid. Since the kinetic energy of each particle is proportional to the square of its velocity, and since this velocity is proportional to the velocity of the solid, the energy of each particle is proportional to the square of the velocity of the solid, and hence the kinetic energy of the fluid as whole. An analogous relation holds for the resulting momentum of the fluid, that is the sum of all momenta of the individual particles. This sum, however, becomes mathematically indefinite, when its evaluation by integration over the space is attempted. The momentum imparted by the force necessary to create the motion can be found, however, and gives the desired quantity. This momentum is proportional to the velocity of the solid, but not necessarily parallel to it.

From the consideration of the energy it follows that the solid, moving with constant velocity, cannot experience a drag, that is, a component of the resulting fluid force parallel to the motion. If it could, there would be a continual consumption of energy without any increase of the energy of the fluid, and this would contradict the principle of the conservation of energy. There may however, exist other components of the resultant fluid force, as will be seen.

Suppose now the motion of the solid to be uniformly accelerated. There is then an external force necessary for the acceleration of the mass of the solid itself. Furthermore, the mass of the fluid has likewise to be accelerated. Although the entire fluid takes part in the motion, particles far away move very slowly and the accelerating force and entire energy developed are both finite. This energy may be written in the form $(1/2)\, A\, U^2$ where A is a constant. This, however, is the expression for the kinetic energy of a solid having the mass A. The kinetic energy and the force necessary to accelerate the solid surrounded by the perfect fluid can therefore be computed by adding to the actual mass of the solid a fictitious mass, called the additional apparent mass of the solid. Since the inertia forces of the fluid are proportional to its density and the latter is assumed constant, this apparent mass is equal to the density of the fluid multiplied by a volume, which volume depends on the geometric outlines of the solid only, including its position relative to the direction of motion. This is called the volume of additional apparent mass.

We proceed to establish an equation for the computation of the kinetic energy of the fluid, to be used for the computation of its apparent mass and the volume of such mass. We suppose the motion to be created

from rest. At the points of the surface of the solid, the pressure term corresponding to the steady motion does not perform any work, as we have just seen, and there remains in II (4.1) only the term $\varrho\,\partial\varphi/\partial t$. This pressure acts along a path equal to the product of the normal velocity component $-\partial\varphi/\partial n$ by the time. With constant acceleration, the term $\partial\varphi/\partial t$ is constant, and hence the potential is proportional to the time t, so that its average value during the interval in which the motion is built up, is half its final value. The energy passed into the fluid through the surface element dS is therefore $-(\varrho/2)\,\varphi\,(\partial\varphi/\partial n)\,dS$ and the entire kinetic energy is the integral of this expression, extended over the entire surface of the solid: thus denoting kinetic energy by T we have,

$$T = -\frac{1}{2}\varrho\int\!\!\int\varphi\,\frac{\partial\varphi}{\partial n}\,dS \tag{2.1}$$

This is the entire energy, because at a great distance there is no motion, and any variation of pressure at such distance cannot give rise to a transfer of energy from distant points.

The same integral (2.1) can be obtained directly by a transformation of the kinetic energy of each particle. This expression can be written as a divergence, and can then be transformed from a space integral into a surface integral by the use of Gauss' transformation I (2.6). Thus let φ and ψ denote any two scalars satisfying II (3.1). Then

$$\nabla\cdot(\varphi\nabla\psi) = \nabla\varphi\cdot\nabla\psi + \varphi\nabla\cdot\nabla\psi = \nabla\varphi\cdot\nabla\psi$$

If then

$$\varphi = \psi$$

$$\nabla\cdot(\varphi\nabla\varphi) = \nabla\varphi\cdot\nabla\varphi$$

Hence

$$\frac{\varrho}{2}\int\nabla\varphi\cdot\nabla\varphi\,dQ = \frac{\varrho}{2}\int\nabla\cdot(\varphi\nabla\varphi)\,dQ = \frac{\varrho}{2}\int\varphi\nabla\varphi\cdot d\boldsymbol{S} \tag{2.2}$$

This is the same result as before.

The volume of apparent mass, more exactly of the additional apparent mass, is obtained by dividing the kinetic energy by the dynamic pressure $U^2\,\varrho/2$ of the velocity of the solid. Thus,

$$K = \frac{2\,T}{\varrho\,U^2} \tag{2.3}$$

This volume can again be divided by the volume of the solid, obtaining then a non-dimensional quantity depending only on the shape of the solid in conjunction with the direction of motion. This number is generally denoted by k, and is called the inertia factor or coefficient of additional apparent mass. The inertia factor becomes infinite, and the definition breaks down if the volume of the solid becomes zero without the kinetic energy becoming zero, as in the case of the motion produced by an infinitely thin disc moving normally to its plane.

Equation (2.1) gives the kinetic energy of the flow for the motion of the solid, with the fluid at rest at large distance. This is the flow unsteady

relative to the fluid at large. It is sometimes more convenient to use the steady motion, with finite and constant velocity at large distance and the solid at rest. The kinetic energy of the entire fluid moving with constant velocity, but for the presence of the solid, is of course infinite, but we are interested in the difference of the kinetic energies of the fluid moving with constant velocity with and without the presence of the solid. We assume again the flow to be built up from rest by constant acceleration. No energy passes through the surface of the solid during the creation of the motion, because its surface does not move. The energy is received by the fluid by means of the pressure variation at a great distance. Hence the integral (2.1) must be extended over a very large surface, for instance, that of a large sphere surrounding the solid.

This integral is somewhat indeterminate, and great care must be taken to approach the limit in keeping with the conditions specified; that is, with the external boundary, such as a sphere, moving like a solid. The method is illustrated by an example in IV 6, and leads there to a very interesting and useful theorem.

It must not be supposed, however, that the kinetic energy obtained from the integral (2.1) applied to the solid surface and applied to the infinite surface is the same. The difference consists of two parts. The first part is the infinite kinetic energy of the fluid due to the velocity at a great distance relative to the solid. The second part is the energy of the fluid replaced by the solid. Eliminating the first part, there will remain a volume of apparent mass larger than the actual one computed by the application of (2.1) to the surface of the solid, and the difference is equal to the volume of the solid itself. The difference of the two corresponding inertia factors is 1.

3. Apparent Momentum. The discussion of the last section resulted in the discovery that a solid surrounded by a perfect fluid moves like a solid in vacuum, except that its mass is apparently increased. There is nevertheless a fundamental difference between the motion of a solid in vacuum and its motion surrounded by a perfect fluid. The mass of the solid in vacuum is equal in all directions. The apparent mass of a solid surrounded by a perfect fluid is in general different in different directions.

Similar arguments as for the translational motion hold for a solid rotating about any axis. The effect of the surrounding perfect fluid will give rise to additional apparent moments of inertia. They are different for different axes, but this is already the case for the moments of inertia in vacuum. The kinetic energy of the motion is again computed by means of the integral (2.1) and the additional moment of inertia is computed from the kinetic energy by dividing it by $\omega^2/2$ where ω denotes the angular velocity.

Turning again to motions of translation, it is not necessary to compute the kinetic energy for each possible motion of the solid separately. If the motion of the solid is the vector sum of two or more other motions, the corresponding velocity distribution can be obtained from the velocity distribution of the component motions by superposition, as this furnishes the flow complying with the boundary conditions. The kinetic energies of the component motions do not, however, add, since they contain the second powers of the velocities. The momenta, however, contain the velocities linearly, and hence they may be superposed the same as the velocities themselves. Since the kinetic energy is the scalar product of the momentum by half the velocity, it is apparent that the energy is a homogeneous function of the second order of the different components of the velocity. The possible motions of a solid are thus determined by six scalar variables, the three components of the velocity, u, v, and w, and the three components of the angular velocity about a specified point of the solid, p, q, and r, all six variables measured by means of a system of coordinates moving with the solid. This gives for the complete expression, homogeneous of second order, six squares, as u^2, q^2, etc., and 15 products of two different factors, as $u\,v$, $w\,q$, etc., in all 21 terms. There are therefore 21 constants depending on the shape of the solid, which give the kinetic energy in connection with the six components of motion. This expression for the kinetic energy with 21 terms can then be used for the computation of the resulting fluid forces, for all possible motions, steady or accelerated.

In most practical cases, a great number of the 21 constants are zero, and then the problem is much simplified. There has never arisen a practical application where all 21 constants had different and finite values. It seems therefore unnecessary to write down the complete mathematical expressions with so great a number of symbols and terms. The general argument can indeed be much better pursued by studying the momentum vector, rather than the kinetic energy.

Let ξ be the momentum and λ the moment of momentum of the fluid motion relative to a reference system moving with the solid, so that the two vectors together represent the efforts required to create the fluid motion. As the solid proceeds, both vectors are assumed to proceed with it, retaining their position relative to the solid, and retaining likewise their magnitude unless the velocity of the solid changes. All velocities, motions, and these two vectors are studied relative to this reference system moving with the solid.

Further, let X denote an external force and L an external moment or couple acting on the solid, defined with respect to the same reference system. With constant velocity U of the solid, the translation of the momentum and of the moment of momentum requires momentum to be continually annihilated at one position and newly created at another

position. This transport of the momentum requires in general an external couple, equivalent to the rate of change of the moment of momentum of the fluid motion. These relations may be expressed in the form

$$\boldsymbol{L} = \frac{\partial \boldsymbol{\lambda}}{\partial t} + \boldsymbol{U} \times \boldsymbol{\xi} \tag{3.1}$$

It follows that a solid moving with uniform translation experiences a moment equal to the cross product of the velocity and the momentum, and hence equal to the component of the momentum at right angles to the motion, multiplied by the velocity. If the motion is accelerated, there is besides, the force $\partial \boldsymbol{\xi}/\partial t$ and the moment $\partial \boldsymbol{\lambda}/\partial t$.

When uniform translation and rotation are combined (steady motion) the equations are

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{\omega} \times \boldsymbol{\xi} + \boldsymbol{X} = 0 \tag{3.2}$$

$$\frac{\partial \boldsymbol{\lambda}}{\partial t} = \boldsymbol{\omega} \times \boldsymbol{\lambda} - \boldsymbol{U} \times \boldsymbol{\xi} + \boldsymbol{L} = 0 \tag{3.3}$$

There exists then a kind of centrifugal force of the apparent tangential mass, if such expression be allowed, and a precessional moment. As the distance of the solid from the assumed axis of rotation becomes larger and larger, and the angular velocity correspondingly smaller, leaving the tangential velocity component constant, the precessional moment vanishes and the centrifugal force likewise, but not its product by the distance from the solid. This distance becomes at last infinitely large and its product by an infinitely small force gives then rise to the couple of the solid moving with translation only.

4. Momentum of a Surface of Revolution. If the solid moving through the perfect fluid has an axis of symmetry, this axis is a principal axis of momentum, and all transverse directions, at right angles to the axis of symmetry, are likewise and in an equal way principal axes. It is therefore sufficient to consider one of them only, together with the axis of symmetry.

Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be parallel to the axis of symmetry and to the lateral direction. Let the velocity be

$$u\boldsymbol{i} + v\boldsymbol{j} = U\,(\boldsymbol{i}\cos\alpha\,\boldsymbol{j} + \sin\alpha) \tag{4.1}$$

where α denotes the angle between the axis and the direction of motion. Let A denote the apparent mass in the direction of the axis of symmetry and B that in the direction of the lateral component motion. Then for the momentum in the field we shall have

$$\boldsymbol{\mu} = A\,U\,\boldsymbol{i}\cos\alpha + B\,U\boldsymbol{j}\sin\alpha \tag{4.2}$$

and for the kinetic energy,

$$T = -\frac{1}{2}\,\boldsymbol{U}\cdot\boldsymbol{\mu} = \frac{1}{2}\,U^2\,(A\cos^2\alpha + B\sin^2\alpha) \tag{4.3}$$

The external moment necessary to maintain the motion steady is

$$\boldsymbol{L} = \boldsymbol{U} \times \boldsymbol{\mu} = (B - A)\,U^2\cos\alpha\sin\alpha\,\boldsymbol{k} \tag{4.4}$$

$$L = \frac{1}{2} U^2 (B - A) \sin 2 \alpha k \tag{4.5}$$

and introducing the volumes of apparent mass, $K_1 = A/\varrho$, $K_2 = B/\varrho$, we obtain

$$L = \frac{\varrho}{2} U^2 (K_2 - K_1) \sin 2 \alpha k \tag{4.6}$$

This is the moment or couple necessary to maintain the motion steady: the fluid moment exerted against the solid is opposite.

We shall see that with an elongated surface of revolution, as the shape of an airship hull, the transverse apparent mass is larger than the longitudinal apparent mass. An axial motion of such hull shaped solid is therefore unstable; the reaction of the fluid tends to increase any small angle between the axis and the direction of motion.

5. Remarks on Lift. It may have surprised the reader to note that the theory of motion of a solid through a perfect fluid fails to account for any lift, that is a resultant force at right angles to the motion, since it is well known that airplane wings experience such lift.

Wings with infinite span, in the two-dimensional problem, make the space doubly connected, and this removes the contradiction between the pure theory and the facts. Compare III 1. Wings of finite span produce a motion of the air not entirely free from rotation. Under the influence of the air friction, there is, in a layer behind the wings, a concentration of vortices. The theory so far developed is well suited for a further extension to take these special conditions into account, and has indeed led to a development of better wing sections and to a better method for performance computations of airplanes. This subject is treated elsewhere in this work.

If there are several solids immersed in the perfect fluid, lift is possible even with irrotational flow. This includes the case where one solid moves through the perfect fluid, contained in a solid container.

CHAPTER IV

SOURCES AND VORTICES

1. Sources and Sinks. We proceed now to take up in detail the study of velocity distributions complying with the conditions of absence of divergence and rotation. The discussion in III 1 referring to the proof of the singleness of the solution shows that there cannot be any motion whatsoever if the divergence and rotation are zero throughout the entire space, and if in addition the velocity approaches zero at large distance. If it approaches constant velocity, the velocity is constant throughout the entire space.

If we suppose one solid at least to be immersed in the fluid, we may imagine the space occupied by the solid to be filled with fluid and the fluid to perform a fictitious motion. It may for instance move like

the solid, or be at rest. This gives rise in general to either rotation or divergence at the surface of the solid, and if the fluid rotates like a solid, this gives rotation through the entire space occupied by the solid. If the fictitious fluid moves like the solid, there is a sudden change of the tangential velocity component at the surface, being equivalent to a distribution of vortex lines over the surface, proportional to the intensity of this sudden change and at right angles to the relative change. If the fictitious fluid rests, there is in general a step in the normal velocity component at the surface, giving rise and being equivalent to concentrated divergence.

It is generally more convenient to specify not a particular state of motion of the fictitious fluid replacing the solid, but to extend the outside flow inside the solid, following the same mathematical laws as outside. The motion cannot be free from divergence and rotation at the same time at all points of the space, but this is not required. It has only to be so in the space not occupied by the solid or solids; that is, in the actual fluid, not in the fictitious one. If in addition the flow complies with the boundary conditions at the surface of the solid, and is of the desired type at large distance, it is the desired flow and constitutes the desired solution outside the solids.

We have therefore to study velocity distributions free from divergence and rotation at nearly but not all points in the space. The simplest case is represented by the absence of rotation at all points, and by the absence of divergence at all points except one, where there is supposed to exist a concentrated divergence. At this point, fluid is continually created, or is entering space from outside. Since the mathematical condition of the flow is entirely symmetrical in all directions, the solution is likewise, and consists of a flow where the fluid moves evenly to all sides along radii passing through the point of divergence. This point, or sometimes the entire flow or velocity distribution under discussion is called a point source or briefly a source. The strength m of the source is measured by the volume of fluid created per unit time, which is identical with the flux through all closed surfaces surrounding the source. Choosing spheres for such surfaces, the velocity at the distance r from the source is seen to be $m/4\pi r^2$. The velocity potential can be obtained by integrating this velocity along one radius thus giving $-m/4\pi r$ plus an integration constant. The velocity is inverse to the square of the distance from the source and vanishes according to that law at a great distance. The flow is not compatible with the fiction of a solid and of an inflexible container of large dimensions surrounding the flow, since such container would require the flux zero rather than a finite flux independent of the shape of the container. The conditions at a great distance, or at infinity, may, however, be assumed as equivalent to a negative source of the same strength, absorbing fluid the equivalent

of that introduced and thus giving a consistent condition of flow. A negative source is known as a sink.

The stream-curves of the source are all radii passing through its center. For drawing stream-lines of unit strength, the surface of the sphere must be divided into m equal parts, and one stream-line associated with each part. There is free choice in this division. We prefer, for reasons that will appear later, to divide the spherical surface into equal parts by means of meridians and parallels, all angles between adjacent meridians being the same, and to draw all radii through these lines. Fig. 4 shows the stream-lines resulting from this choice, in a plane through the axis connecting the poles. These stream-lines have not equal angular difference, but the difference between the cosines of the angles of adjacent stream-lines with the axis is constant. The surface of the portion of a sphere between two parallel planes is proportional to the distance between these planes. The stream-line diagram is therefore different from that for the corresponding plane two-dimensional flow.

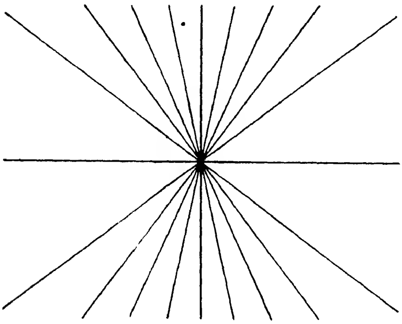

Fig. 4.

2. Superposition of Two Sources. The advantage of this particular method of selecting and drawing the stream-lines becomes apparent if we proceed to the case of two sources, or a pair comprising a source and a sink, and superpose two such flows. The resulting flow being symmetrical about the line of connection of the two sources as axis, we choose this line as axis of the diagram.

The superposition of two flows can be performed graphically, by drawing first the two systems of stream-lines to be superposed. The two systems intersect each other forming quadrilaterals, and the stream-lines of the flow resulting from the superposition form one set of diagonals of these quadrilaterals, in the same general manner as in the case of two-dimensional flow. See Fig. 5.

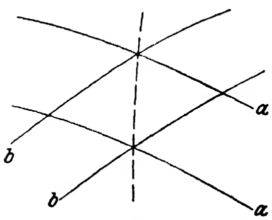

Fig. 5.

The method holds only if crossing from one corner of the quadrilateral to the other corresponds to a constant and equal change in the flux in each component system, increase for one and decrease for the other. This is the case with the present specification, since all stream-tubes are bounded by the same plane through the axis. The same method can be employed for drawing the lines of equal potential of a flow

resulting from the superposition of two other flows, the lines of equal potential of which are drawn. The proof is the same, since the potentials superpose like the flux. The method can always be employed with the potential, since the value of the potential at each point is always defined in a potential flow. The value of the flux or stream function, as it is called, is not defined except in special cases like the present one, where one set of surfaces being occupied by stream-lines is easily detected, in this case from considerations of symmetry[1].

Fig. 6 shows the flow resulting from the superposition of one source and one sink of equal strength. All stream-lines start then at the source and end at the sink. Fig. 7 shows the parallel and constant flow drawn the same way. The flow has to be parallel to the axis, as otherwise it would not be symmetric with respect to it. The stream-lines are not equidistant, but the volume between the cylinders represented by them is constant.

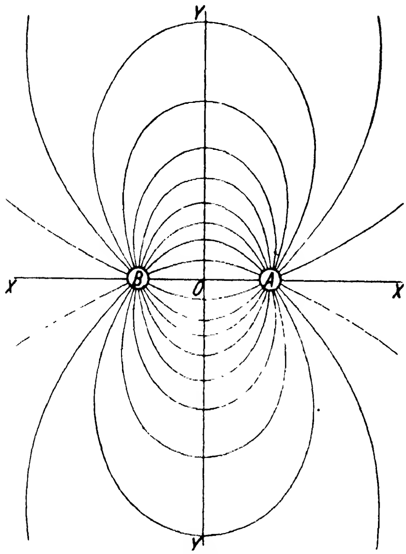

Fig. 6.

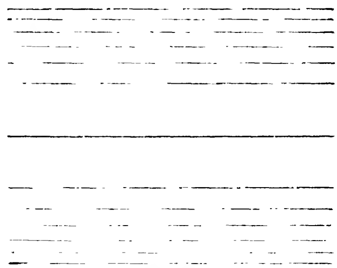

Fig. 7.

3. Doublet. As a special case of the flow resulting from the superposition of a source and a sink of equal strength we assume the source and sink to approach each other very closely. Their effect on an even slightly distant point is then neutralized, and there would not remain any motion. However, we specify further, that the strength of both, source and sink, shall be chosen so that the product of strength and distance shall remain constant as the two approach each other. As the distance becomes very small, there will remain a definite velocity distribution, independent of the nearness of approach, so long as it is small. Such special kind of superposition is equivalent to a differentiation. Let the distance between the source and sink be dx and M the constant

[1] Division B IV 1 and B IX 1.

product of strength and distance. Then the two strengths will be M/dx and $-M/dx$. The combined potential will then be[1]

$$\varphi_1 + \varphi_2 = \frac{1}{dx}\left[\frac{M}{4\pi\sqrt{(x-dx)^2+y^2+z^2}} - \frac{M}{4\pi\sqrt{x^2+y^2+z^2}}\right]$$

This, however, is readily seen to give

$$\varphi = -\frac{\partial}{\partial x}\frac{M}{4\pi\sqrt{x^2+y^2+z^2}} = -\frac{\partial}{\partial x}\frac{M}{4\pi r} \tag{3.1}$$

where r denotes $\sqrt{x^2+y^2+z^2}$. This value of φ readily transforms to

$$\varphi = \frac{Mx}{4\pi r^3} = \frac{M\cos\theta}{4\pi r^2} \tag{3.2}$$

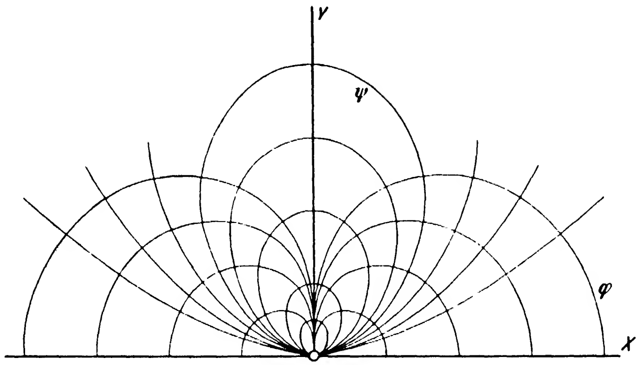

Fig. 8.

This special superposition by differentiation can always be employed. For, if φ complies with Laplace's equation, so also do the differential quotients of φ with respect to distance, such as $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, and $\partial\varphi/\partial z$, and hence any sum of the three multiplied by arbitrary constants $-l\,\partial\varphi/\partial x - m\,\partial\varphi/\partial y - n\,\partial\varphi/\partial z$. If l, m, n, are the direction cosines of a given line or direction in space, this expression gives the rate of increase of φ along this line or direction.

The stream-lines of a doublet are shown in Fig. 8 with the axis directed from left to right. All stream-lines begin and end in the same point.

4. Polar Coordinates. The remaining part of this monograph will consist, in large part, of discussions of the solutions of Laplace's equation

[1] It is obvious that the potential φ may be taken in either the positive or negative sense with regard to differentiation along the axes of x, y and z. In the present Division it is so taken that the negative derivative along the axes of x, y and z gives the component velocities u, v and w along these axes; or in general, that the velocity in any direction is given by the negative derivative of φ in that direction. In other Divisions of this work, the reverse will usually be found. Naturally the interpretation of final results is the same in either case. (Note by Editor.)

representing potential flows having some bearing on aeronautic problems. The theory is not difficult, but it is involved. One of the main difficulties, as always in involved mathematics, lies in grasping and memorizing the exact meaning of all symbols occurring.

As will be seen, the solutions are obtained by selecting a suitable system of coordinates, and by expressing the condition of continuity in terms of these coordinates. We have thus far employed the Cartesian system, x, y, and z, at right angles to each other. The surfaces, $x =$ *constant*, $y =$ *constant*, and $z =$ *constant* are planes at right angles to the axes of the system of coordinates, and are therefore mutually at right angles to each other.

In addition to this system, it will be found needful to use systems where at least one of the three families of surfaces determined by one of the coordinates being constant, consists of curved surfaces. All systems to be employed, however, retain the property of orthogonality, possessed by the Cartesian coordinates. The three surfaces representing a constant coordinate will always intersect with each other at right angles. Orthogonal systems only have proven to lead conveniently to solutions of Laplace's equation.

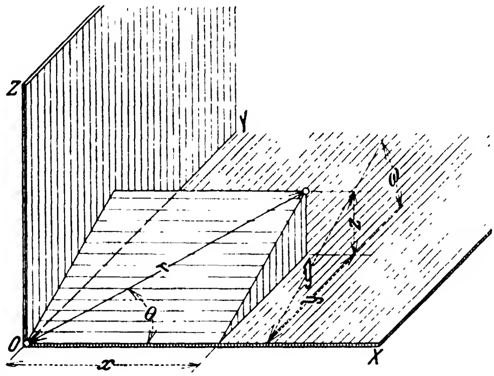

Fig. 9.

In the following chapters, use will be made of a system of polar coordinates. We obtain them by starting with Cartesian coordinates x, y, and z, all of which represent lengths. Two systems of polar coordinates will be used, one using two lengths and one angle, and the other using one length and two angles, Fig. 9. The former stands halfway between Cartesian coordinates and polar coordinates in its strictest meaning, and this system will therefore be called semi-polar coordinates. We specify in either case the x axis as the axis of the new systems, and consider likewise the origins of all three systems to coincide. In the semi-polar system, x remains as a linear coordinate, continuing to be the distance of the point in question from the y z plane through the origin. In the x y plane through the origin, but only there, y continues likewise to have its old meaning. In the semi-polar system, we introduce as second coordinate, the distance from the x axis, and denote it by $\bar{y}$ in order to remind the reader that this coordinate denotes a length and agrees with y in the plane $z = 0$.

The third coordinate in the semi-polar system is an angle, denoted by ω. This is the axial angle between two planes passing through the x axis, the one plane being $z = 0$ and the other plane containing the point in question.

The Cartesian coordinates x, y, and z, and the semi-polar coordinates x, $\bar{y}$, and ω are connected by means of the following equations:

$$\left.\begin{aligned} x &= x \\ \bar{y} &= \sqrt{y^2 + z^2} \\ \omega &= \tan^{-1} \frac{z}{y} \end{aligned}\right\} \qquad (4.1)$$

The equation $\bar{y} = constant$ represents a cylinder around x as axis. The equation $\omega = constant$ represents a plane through the x axis. We call ω the axial angle because it can be measured at all points of the axis.

Polar coordinates have the angle ω in common with semi-polar coordinates, both with respect to notation and meaning. In each plane through the x axis, the two length coordinates x and $\bar{y}$ are replaced by one length, r, the distance from the origin, and the angle θ between the x axis and the line connecting the point in question with the origin. The equation $\theta = constant$ represents therefore a cone around x as axis. This is sometimes called the apex angle, being measured at the apex of the cone at the origin.

It is at once apparent, that in the polar system the two angles ω and θ are by no means symmetrical and cannot be exchanged with each other, the one, ω, being constant for planes and the other, θ, for cones co-axial with x. The equation $r = constant$ represents spheres with the origin as center.

The mathematical relations between the semi-polar coordinates ω, x, and $\bar{y}$, and the polar coordinates follow directly,

$$\left.\begin{aligned} \omega &= \omega \\ r &= \sqrt{x^2 + \bar{y}^2} \\ \theta &= \tan^{-1} \frac{\bar{y}}{x} \end{aligned}\right\} \qquad (4.2)$$

It remains finally to write the transformation equations from the Cartesian to the polar coordinates directly. This is done by combining the two sets of transformation equations, (4.1) and (4.2), eliminating the semi-polar coordinates. There results

$$\left.\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta \cos \omega, \\ z &= r \sin \theta \sin \omega \end{aligned}\right\} \qquad (4.3)$$

5. Motion of a Sphere. Fig. 10 shows the superposition of a doublet $\varphi = \frac{M}{4\pi r^2} \cos \theta$ and of the constant velocity $\varphi = U x = U r \cos \theta$. It is seen that a surface, apparently spherical, divides the flow into two

separate parts, the outside being free from singularities and representing the flow around the surface. The potential of the combined flow is

$$\varphi = U r \cos\theta + \frac{M}{4\pi r^2}\cos\theta \tag{5.1}$$

We verify the spherical shape of the surface by computing the components of the velocity represented by the potential (5.1) normal to the surface of a sphere with the radius r. This velocity is

$$-\frac{\partial\varphi}{\partial r} = -U\cos\theta + \frac{M}{2\pi r^3}\cos\theta$$

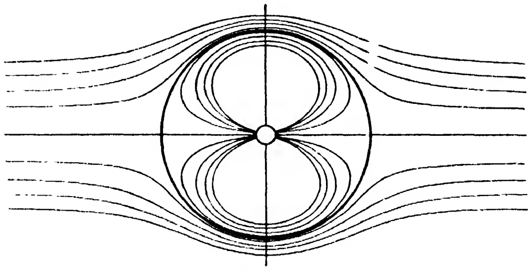

Fig. 10.

This becomes zero for $M = +2\pi r^3 U$. Denote this particular value of r by R and thus numerically

$$R = \sqrt[3]{\frac{M}{2\pi U}}$$

$$M = 2\pi U R^3$$

This gives for the potential (5.1)

$$\varphi = U\cos\theta\left[r + \frac{R^3}{2r^2}\right] \tag{5.2}$$

The potential of the flow relative to the fluid at a great distance is then,

$$\varphi = U\cos\theta\,\frac{R^3}{2r^2} \tag{5.3}$$

At the intersection of the axis and the sphere the velocity relative to the sphere becomes zero. Such point of zero velocity is called the impact point or point of stagnation. The tangential velocity from (5.3) is

$$-\frac{1}{R}\frac{\partial\varphi}{\partial\theta} = \frac{1}{2}U\sin\theta \tag{5.4}$$

The velocity along the meridian is therefore proportional to the distance from the axis X. The pressure distribution over the surface is computed from (5.4) in combination with II (4.2) and results

$$p = -\frac{\varrho}{2}U^2\cdot\frac{1}{4}\sin^2\theta$$

measured from the pressure at the points of stagnation, or

$$p = \frac{\varrho}{2}U^2\left(1 - \frac{1}{4}\sin^2\theta\right)$$

measured from the pressure at . . p distance, relative to which the pressure at the point of stagna . . . equal to the dynamic pressure of the motion. Equation (5.2) wr e in Cartesian coordinates becomes

$$\varphi = Ux\left[\,\cdot + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

Exchanging herein x and y, that s turning the flow relative to the system of coordinates, gives

$$\varphi = Uy\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

and going now back to polar coordinates, (4.3) gives

$$\varphi = U\,r\,\sin\theta\,\cos\omega\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right] \tag{5.5}$$

The velocity component tangential to circles in planes parallel to the motion and having a radius $r \sin\theta$ is

$$-\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\omega} = U\sin\omega\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right] \tag{5.6}$$

It appears therefore that this velocity component, which may be called peripheral velocity component, is likewise proportional at all points of the same circle, to the distance from a diameter.

The maximum velocity takes place at the intersection of the sphere with the plane of symmetry at right angles to the motion. This maximum velocity is 1.5 U relative to the sphere, and 1/2 U relative to the fluid at rest at a large distance from the sphere.

The flux through the circle containing a point with the distances $\bar{y}$ from the axes and r from the origin is

$$\pi\,U\,\bar{y}^2\left(1 - \frac{R^3}{r^3}\right)$$

It is at last interesting to compute the resultant air force acting on each *half* sphere. It is obtained by integrating the pressure multiplied by the projections of the surface elements in the direction of the axis. The integral becomes

$$-\frac{2\pi\,\frac{\varrho\,U^2}{2}}{4\,R^2}\int_0^R \bar{y}^3\,d\,\bar{y}$$

giving a suction force $-\frac{1}{8}\pi R^2 U^2 \frac{\varrho}{2}$ measured from the impact pressure. It is plausible that this force should be a suction and not a pressure. The fluid when it reaches the plane of symmetry at a right angle to the motion, has attained a velocity opposite to the motion of the sphere, requiring a suction rather than a pressure for its production.

6. Apparent Mass of the Sphere. We compute the kinetic energy of the flow (5.3) using III (2.1)

$$T = \frac{\varrho}{2}\int \frac{U}{2} R \cos\theta \, U \cos\theta \, 2\pi R^2 \sin\theta \, d\theta = \frac{2}{3}\pi R^3 \frac{\varrho U^2}{2} \qquad (6.1)$$

Hence, the volume of apparent additional mass is equal to half the volume of the sphere, and the inertia factor of the sphere is 1/2.

Applying III (2.1) to the flow, (5.2) gives infinite energy of the fluid filling the entire space and having the constant velocity U, diminished by a certain amount, which we will now compute. This requires the assumption of a rigid spherical surface with the radius Q surrounding the flow. The potential is of the form

$$\varphi = \cos\theta\left(\frac{A}{r^2} + B r\right)$$

$$\frac{\partial\varphi}{\partial r} = \cos\theta\left(-\frac{2A}{r^3} + B\right)$$

giving
$$\cos\theta\left(-\frac{2A}{r^3} + B\right) = 0$$

at the surface of the solid sphere and

$$\cos\theta\left(-\frac{2A}{r^3} + B\right) = U\cos\theta$$

at the surface of the large sphere. This is satisfied by

$$A = \frac{1}{2}\,\frac{U R^3}{1 - \left(\frac{R}{Q}\right)^3} \qquad B = \frac{U}{1 - \left(\frac{R}{Q}\right)^3}$$

The integral is now

$$T = \frac{\varrho}{2}\int 2\pi Q^2 \sin\theta\, d\theta\, U\cos\theta\, U\cos\theta \left[\frac{R^3}{2Q^2\left[1 - \left(\frac{R}{Q}\right)^3\right]} + \frac{Q}{1 - \left(\frac{R}{Q}\right)^3}\right]$$

The bracket can be written

$$\frac{\left(\frac{1}{2}R^3 + Q^3\right)}{Q^2\left[1 - \left(\frac{R}{Q}\right)^3\right]} \sim \frac{1}{Q^2}\left[\frac{R^3}{2} + R^3 + Q^3 + \ldots\right] \qquad (6.2)$$

The last term gives the infinite kinetic energy for the constant and parallel flow. The other two terms give

$$1.5\left(\frac{4}{3}\pi R^3\right)\frac{\varrho U^2}{2}$$

This is 1.5 of the volume of the sphere, leaving for the inertia factor of the flow outside the sphere $1.5 - 1 = 0.5$, see III 2. This agrees with the previous result.

7. Apparent Mass of Other Source Distributions. In anticipation of later use, we compute the kinetic energy of the flow having as the only singularity a source and sink of equal strength. Let the strength be m, and their distance be a, parallel to the constant velocity U of motion of the fluid at a great distance.

Let the source be at the origin. Its potential is $m/4\pi r$. The potential of the sink, with the strength $-m$ is

$$\frac{-m}{4\pi\sqrt{(x+a)^2+y^2+z^2}} = \frac{-m}{4\pi\sqrt{r^2+2xa+a^2}} \tag{7.1}$$

The potential of the two sources, combined, is therefore

$$\frac{m}{4\pi r} - \frac{m}{4\pi\sqrt{r^2+2ax+a^2}}$$

We apply now the binomial development to the potential of $-m$, on the assumption that r is large compared with a. This gives

$$-\frac{m}{4\pi\sqrt{r^2+2ax+a^2}} = \frac{-m}{4\pi r}\left[1-\frac{ax}{r^2}+\ldots\ldots\right]$$

Hence the potential of the superposed flow can be developed in the form

$$\frac{m}{4\pi}\frac{x}{r^3}a+\ldots\,.$$

Putting this into the integral III (2.1) we see that only the term $\frac{m}{4\pi}\frac{ax}{r^3}$ contributes to its value as r becomes infinitely large.

This is the potential for a doublet, as might be expected, at a distance r, large compared with a, the distance between the source and the sink. Then from (6.1) we have the kinetic energy for a sphere of radius R and substituting therein the value of $R^3 = M/2\pi U$, we have $T = \varrho UM/6$. If then we put $T = K\varrho U^2/2$, we shall find for K, the volume of apparent mass, the value $\qquad K = \frac{M}{3U} = \frac{ma}{3U}$

If there are more than one pair of source and sink, any number of them, such that the sum of their intensities is zero, they can be arranged into pairs of equal strength, and the preceding calculation can be applied to each pair. It follows therefore that the apparent mass volume of the solid having these pairs as a fictitious flow is equal to the summation of a series of terms each of the form ma, the whole divided by $3U$ thus,

$$K = \frac{\Sigma ma}{3U} \tag{7.2}$$

8. Vortices. Further discussion of the flow produced by moving solids may be deferred to the following chapter. We now pass to some discussion of the other singularity that may occur, concentrated vorticity.

The simplest case is a straight vortex line of infinite strength, so that the scalar line integral around it, the so-called circulation, has a finite value, independent of the shape of the integration path. This gives the so-called strength of the vortex. All points not on the vortex line are supposed to have rotation zero, and paths for the computation of the circulation must surround the vortex once only.

Reference to Division B III 2 will show that the existence of any line vortex of the character assumed requires that there be associated

at a point P, with vortex elements of the length ds at a point Q, velocity elements with the magnitude $dw = \frac{\Gamma \, ds \sin \theta}{4 \pi r^2}$ (8.1)

where r is the distance PQ and Θ is the angle which PQ makes with the vortex line. These velocity elements are at right angles to PQ and to the vortex element. The velocity distribution associated with a distribution of vortex lines can be computed by integrating this expression through all vortex lines. Limiting the integration to fragments of vortex lines does not yield an irrotational flow.

This integration is more perfectly expressed in vector language. The velocity w is obtained from the vorticity Γ by combining it with the vector integration operator

$$Int = \frac{1}{4\pi} \int d\,S' \frac{(r - r')}{[(r - r') \cdot (r' - r)]^{3/2}} \tag{8.2}$$

by cross multiplication, thus

$$w = Int \times \Gamma = \frac{1}{4\pi} \int d\,S' \frac{(r - r') \times \Gamma}{[(r - r') \cdot (r' - r)]^{3/2}} \tag{8.3}$$

In these expressions r is the radius vector of the point for which the velocity is computed, r' is the radius vector of the vortex elements, and the integration extends over all of them through the entire space S'.

Applied to a straight line vortex, this integral gives a velocity at right angles to the vortex and inverse in magnitude to the distance from the vortex line.

9. Forces Between Sources and Vortices. We conclude this chapter by establishing rules for the computation of the resultant fluid forces, if the fictitious sources and sinks are known.

These rules are based on II (5.3) referring to the forces on regions containing sources or vortices. The sources are pulled in the direction of the velocity of the region, the sinks are repelled opposite and the vortices undergo side or lateral forces.

Applying this to the straight motion of one solid, the outside motion is merely the constant velocity U. This cannot exert any resultant force, since the intensity of all sources and sinks when added up, is zero. Otherwise there would be a flux through the surface and this would constitute a contradiction. If the sum of the intensities of the sources and sinks is zero, the sum of their forces is likewise zero.

Not so the moment. If it is zero, the solid moves in a principal direction of the momentum dyadic, and if the volume of the apparent mass in that direction is K_1 the static moment of the sources for this motion will be $K_1 U_1/4\pi$. For the motion at right angles, the static moment of the sources will be $K_2 U_2/4\pi$. For a superposition of these two flows, representing a motion oblique to a principal direction at the angle φ, the sources of the one component of the motion combined with the

motion in the other principal direction gives rise to forces resulting in the moment

$$K_1\, \varrho\, U \cos \varphi\, U \sin \varphi \tag{9.1}$$

and the other sources with the velocity in the first direction similarly, resulting in the moment

$$-K_2\, \varrho\, U \sin \varphi\, U \cos \varphi \tag{9.2}$$

The entire resulting moment or couple is therefore

$$(K_1 - K_2)\, U^2 \frac{\varrho}{2} \sin 2\varphi \tag{9.3}$$

which agrees with III (4.6).

If there are several bodies, there are the effects of the fictitious sources and sinks within one body on those within the others. Since the velocity of a source is inverse to the square of the distance, the forces between two sources are likewise inverse to the square of the distance. The magnitude of the force between two sources with the intensity m_1 and m_2 and the distance r is therefore

$$\frac{m_1 m_2}{4 \pi r^2}$$

There is attraction between two sources or two sinks, but repulsion between a source and a sink.

The same argument leads to an attraction between two parallel vortex elements, and to a repulsion if they are anti-parallel. In case they are oblique, the expression must be multiplied by the cosine of their angle, so that the force between two vortex elements of the length L_1 and L_2 with circulation Γ_1 and Γ_2 the distance r and the angle φ is

$$\frac{L_1 L_2 \Gamma_1 \Gamma_2}{4 \pi r^2} \cos \varphi \tag{9.4}$$

It is also interesting to apply the formula for the force of vortices to the case where the fluid is moving as a solid body, and the surface is covered with vortices of density equal per unit area to the velocity of the fluid V, relative to the solid. Multiplying this strength by the velocity of the region, which on the average is $(1/2)\, V$, the vortex being surrounded by fluid with the velocity V on one side and by resting fluid on the other side, gives $(1/2)\, \varrho\, V^2$. This agrees with the expression for the pressure in Bernoulli's formula, II (4.2).

CHAPTER V
FLUID MOTION WITH AXIAL SYMMETRY

1. Stream Function. We proceed to discuss in this chapter the irrotational motion of a perfect fluid possessing an axis of symmetry. All flows discussed thus far in detail are of this kind—a constant velocity flow parallel to the axis, a source located on the axis and a doublet with axis parallel to the axis and located on it. A great variety of axially symmetric flows may be built up of these elements by plain superposition and by differentiation in a direction parallel to the axis.

Since the distribution of velocity is the same in all planes passing through the axis, and since the velocity components at right angles to these planes are zero, it is sufficient to study the distribution in one such plane only, and to consider the flow pattern in such plane as representative of the entire flow. It is sufficient to study the velocity distribution in two dimensions only, and in this sense the axially symmetric flow may be called a two-dimensional flow. One series of congruent surfaces, planes in this case, are known from the beginning to be parallel to the stream-lines, and to have congruent stream patterns. If the axis is very far away, the planes approach parallelism and we obtain the case of the plane two-dimensional flow. It is this special two-dimensional flow which is generally referred to by the expression "two dimensional flow", the term "plane" being then implied. We shall here discuss the more general case of the two-dimensional flow with axial symmetry.

We picture the flow again by means of stream-lines, which are arranged as in IV 1 but have a slightly different meaning. We consider any line drawn in an axial plane to represent the area of the surface of revolution of which it is the meridian, divided by 2π. Any element ds of such a line will represent therefore a surface of area $\bar{y}\, ds$, where $\bar{y}$ denotes the distance from the axis. Hence the flux through this surface will equal the product of the velocity and the line element multiplied by the distance $\bar{y}$ and by the sine of the angle between the direction of the velocity and of the element ds[1]. For brevity this may be termed the flux of the line element and the flux across any curve will be, then, the sum of the fluxes of its elements.

Consider now a closed curve. If there are no regions of divergence inside the surface of revolution of which it is the meridian, its flux is zero. Hence the same arguments as used in I 3 lead to the conclusion that the flux of all curves connecting a point with the origin is the same, and can be considered as a scalar function of this point. This flux is then a function, similar to the potential, called the *stream function*, a scalar quantity depending on the space coordinates. This can be used similarly as the potential for computing the velocity components.

Denoting this function by ψ, it follows from its definition,

$$u = -\frac{\partial \varphi}{\partial x} = -\frac{1}{\bar{y}}\frac{\partial \psi}{\partial \bar{y}} \qquad v = -\frac{\partial \varphi}{\partial y} = \frac{1}{\bar{y}}\frac{\partial \psi}{\partial x} \tag{1.1}$$

The velocity is at right angles to the gradient of the stream function and hence the lines of constant potential and the lines of constant stream function are at right angles to each other. These latter lines are the stream-lines themselves, for obviously the flux along a stream-line

[1] Note will be taken of the special unit here introduced for the measure of the stream function ψ, resulting in the omission of the factor 2π found in the corresponding discussion in Division B IX 1.

is zero and hence does not change the stream function when proceeding along the line.

2. Equation of Continuity. The differential equation for φ in the axially symmetric flow can be obtained either by transforming Laplace's equation, II (3.1) or directly from the condition of absence of divergence or flux through a small closed surface. Both methods are simple, the first does not require any new geometrical relations but some familiarity with partial differentiation. The second method is the more direct.

We suppose the x axis to coincide with the axis of symmetry and leave this coordinate unchanged, that is we employ semi-polar coordinates. $\bar{y}$ is the distance from the axis, and ω the angle between the plane passing through the point and the x axis and the $x\,y$ plane, so that we have

$$\bar{y}^2 = y^2 + z^2 \qquad \tan\omega = \frac{z}{y} \tag{2.1}$$

We express now the flux through the ring having as meridian curve a small rectangle with the sides $d\,x$ and $d\,\bar{y}$. Denoting the axial and radial components of the velocity by u and $\bar{v}$, we have

$$u = -\frac{\partial\varphi}{\partial x} \quad \text{and}$$

$$\bar{v} = -\frac{\partial\varphi}{\partial\bar{y}}$$

We obtain the net flux into the annular space through the two pairs of faces which correspond to $d\,x$ and $d\,\bar{y}$.

$$2\pi\,\bar{y}\left[\frac{\partial u}{\partial x} + \frac{\partial\bar{v}}{\partial\bar{y}}\right] d\,x\,d\,\bar{y}$$

To this must be added the difference in flux across the inner and outer faces corresponding to $d\,x$, due to the difference in area. This will be $2\pi\,\bar{v}\,d\,\bar{y}\,d\,x$. Adding these and dividing by the volume of the annulus, $2\pi\,\bar{y}\,d\,\bar{y}\,d\,x$, we shall have for the divergence

$$-\left[\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial\bar{y}}\left(\frac{\partial\varphi}{\partial\bar{y}}\right) + \frac{1}{\bar{y}}\left(\frac{\partial\varphi}{\partial\bar{y}}\right)\right]$$

The equation of continuity in semi-polar coordinates for axial symmetry then becomes

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial\bar{y}^2} + \frac{1}{\bar{y}}\frac{\partial\varphi}{\partial\bar{y}} = 0 \tag{2.2}$$

The differential equation for the stream function cannot be obtained from the condition of zero divergence, as the existence of a stream function implies this already. It does not imply absence of rotation, and this condition gives the desired differential equation. From the definition of the stream function we have

$$u = -\frac{1}{\bar{y}}\frac{\partial\psi}{\partial\bar{y}} \qquad \bar{v} = \frac{1}{\bar{y}}\frac{\partial\psi}{\partial x} \tag{2.3}$$

and the condition of zero rotation is

$$\frac{\partial u}{\partial \bar{y}} - \frac{\partial \bar{v}}{\partial x} = 0 \text{ [1]}$$

Substituting herein (2.3) gives

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{\bar{y}} \frac{\partial \psi}{\partial \bar{y}} = 0 \qquad (2.4)$$

It results that the two functions, the potential φ and the stream function ψ are governed by different differential equations, and hence cannot exchange their role. Only if $\bar{y}$ becomes infinite, the motion approaching the plane two-dimensional flow, the two terms

$$\frac{1}{\bar{y}} \frac{\partial \varphi}{\partial \bar{y}} \quad \text{and} \quad -\frac{1}{\bar{y}} \frac{\partial \psi}{\partial \bar{y}}$$

become zero and the two equations, (2.2) and (2.4), become the same. In that special case only, the potential and the stream function are conjugate and can exchange their roles.

3. Zonal Spherical Harmonics. We proceed next to a short discussion of another method available for obtaining certain solutions of Laplace's equation II (3.1), when it represents potential flows with axial symmetry. This is the use of the so-called zonal spherical harmonics.

Spherical harmonics are solutions of $\nabla^2 \varphi = 0$ which are homogeneous in the Cartesian space coordinates x, y, and z, so that they may be written as the sum of a series of terms

$$\varphi = \Sigma\, x^\alpha y^\beta z^\gamma \qquad (3.1)$$

where the sum of $\alpha + \beta + \gamma$ is equal for all terms.

There exists an extensive mathematical literature on these functions.

For present purposes, consideration may be limited to the special case where all exponents α, β, and γ are positive or negative integers.

Each coordinate x, y, and z can be expressed by the product of r^n, (r = distance from the origin) multiplied by a function of the direction of the line connecting the point with the origin. Hence these functions may be put in the form $\quad \Phi_n = r^n S_n$

where S_n is a function of the direction of the radius only. At points on the unit sphere about the origin, $r = 1$, the spherical harmonic becomes directly $\quad \Phi^n = S_n$

S_n is called a surface harmonic.

The present discussion is limited to axial symmetry and the axis is taken coincident with X. All such spherical harmonics that are finite over the unit sphere and at infinity can then be obtained by repeatedly differentiating $1/r$ with respect to dx. The differentiation is simplified by the use of the relation $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

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[1] Division A VII (3.2).

The following spherical harmonics are obtained in this way.

$$\left.\begin{aligned}\varphi_{-1} &= \frac{1}{r}\\ \varphi_{-2} &= \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3} = -\frac{x}{r}\cdot\frac{1}{r^2}\\ \varphi_{-3} &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3\,x^2}{r^5}\end{aligned}\right\} \tag{3.2}$$

Multiplying these functions by certain constants, and writing for $\cos\theta$

$$\mu = \frac{x}{r}$$

the surface harmonics are generally denoted by the letter P, and become

$$\left.\begin{aligned}P_0(\mu) &= 1\\ P_1(\mu) &= \mu\\ P_2(\mu) &= \frac{1}{2}(3\,\mu^2 - 1)\end{aligned}\right| \tag{3.3}$$

All of these harmonics correspond to combinations of sources and sinks of infinite strength arranged along the x axis in close vicinity of the origin. φ_{-1} is one source, φ_{-2} the doublet, φ_{-3} four sources, etc.

4. Differential Equation for Zonal Surface Harmonics. By virtue of the axial symmetry, the zonal surface harmonics are functions of one variable only, the angle θ between the axis and the radius vector, or of μ the cosine of the angle. We proceed now to establish the differential equation of the zonal surface harmonics expressed in terms of μ. We introduce first the polar coordinates

$$\left.\begin{aligned}x &= r\cos\theta & \bar{y} &= r\sin\theta\\ y &= \bar{y}\cos\omega & z &= \bar{y}\sin\omega\end{aligned}\right| \tag{4.1}$$

giving $\quad x = r\cos\theta;\quad y = r\sin\theta\cos\omega;\quad z = r\sin\theta\sin\omega.$

Giving due regard to axial symmetry, we express the net flux through the surface of a small volume element

$$r\,\delta\theta\cdot r\sin\theta\,\delta\omega\cdot\delta r$$

The flux through the pair of surfaces at a right angle to r is

$$\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\omega\right)\delta r$$

the flux through the pair passing through the origin

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{r\,\partial\theta}\cdot r\sin\theta\,\delta\omega\,\delta r\right)\delta\theta$$

and the flux through the pair passing through the axis is zero in view of the symmetry of the flow.

The addition gives

$$\sin\theta\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\varphi}{\partial\theta}\right) = 0 \tag{4.2}$$

We introduce now $\varphi = r^n S_n$ into the foregoing equation of continuity and carrying out the operations indicated, we find

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d S_n}{d\theta}\right) + n\,(n+1)\,S_n = 0$$

or, introducing

$$\mu = \cos\theta$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\,\frac{d S_n}{d\mu}\right] + n\,(n+1)\,S_n = 0 \tag{4.4}$$

We know already some solutions of this equation, *viz.*, the functions (3.3). Other solutions can be obtained by direct integration. Put $n = 0$. A first integration gives then

$$(1-\mu^2)\,\frac{d S_0}{d\mu} = \text{const.}$$

or

$$\frac{d S_0}{d\mu} = \frac{\text{const.}}{1-\mu^2} = \text{say}\ \frac{1}{1-\mu^2}$$

Integrating this again gives

$$S_0 = \int \frac{d\mu}{1-\mu^2} = \frac{1}{2}\log\frac{1+\mu}{1-\mu}$$

This solution corresponds to a distribution of sources all along the axis. The corresponding solutions for n different from 0 are generally denoted by Q. We have

$$\left.\begin{aligned} Q_0 &= \frac{1}{2}\log\frac{1+\mu}{1-\mu} \\ Q_1 &= \frac{1}{2}\,\mu\log\frac{1+\mu}{1-\mu} - 1 \\ Q_2 &= \frac{1}{4}\,(3\mu^2-1)\log\frac{1+\mu}{1-\mu} - \frac{3}{2}\,\mu \end{aligned}\right| \tag{4.5}$$

The correctness of these solutions is verified by substituting them into (4.4). They are real for $\mu < 1$. Solutions real for $\mu > 1$, not identical with the former ones, are

$$\left.\begin{aligned} Q_0(\zeta) &= \frac{1}{2}\log\frac{\zeta+1}{\zeta-1} \\ Q_1(\zeta) &= \frac{1}{2}\,\zeta\log\frac{\zeta+1}{\zeta-1} - 1 \end{aligned}\right| \tag{4.6}$$

Remarks on the Application of Zonal Harmonics. A brief discussion of zonal harmonics has been included for the reason that use will be found for them at a later point. They are useful in many branches of theoretical physics, because every function axially symmetrical over the surface of the unit sphere can be expanded in terms of these zonal harmonics, similar to a Fouriers series, and the coefficients can be determined in the same way, by integration. In this manner, linear differential equations that are solved by each individual harmonic can be solved for arbitrary symmetrical boundary conditions over the sphere. In the same manner the more general harmonics, referred to later, can be used for the solution of problems not axially symmetric.

For problems arising in aeronautics, the motion of the solid sphere only, belongs to that class, and this problem has already been discussed in IV 5. The solution was found to be the doublet, which is indeed a zonal harmonic. For a more extended discussion of these functions, reference should be made to the mathematical literature on the subject.

5. Superposition of Sources. Another way of obtaining stream-lines around a solid with axial symmetry consists in combining any distribution of sources and sinks along a finite length of the axis, the strength of the sources and sinks varying continuously or discontinuously, but so that the sum of all strengths is zero. If the strength is denoted by $m = f(x)$ the same velocity distribution can also be represented by the distribution of doublets of strength $\int m\,dx$ parallel to the axis.

With complicated source distributions, the contour of the solid and of the stream-lines are obtained by graphical methods, by means of which all sources and sinks are superposed upon each other and then combined with a constant flow parallel to the axis. The contour is the only closed stream-line in the system. The curves of constant stream function are plotted, and this is most conveniently done by plotting first curves of constant x in a ψ diagram. With some practice, and with suitable choice of the distribution of the sources and sinks, it is possible to obtain contours having certain desired characteristics. A concentration of the sources will result in a swelling out of the contour at that location. This method does not, however, constitute a direct way for obtaining the stream-lines or velocity distribution for a given contour. A direct and strict solution of that problem, other than by trial and error methods, has not, so far been devised.

The pressure distribution over the surface is computed from II (4.2) using the velocity distribution as determined. The stagnation point is always at the intersection of the contour with the axis. The pressure there is equal to the dynamic pressure of the motion.

An essential defect of the method is the absence of a way for obtaining the axial flow and the lateral flow for the same contour. The solution is incomplete because it is restricted to the angle of attack zero.

Not every surface of revolution can be replaced with respect to its production of fluid motion by a distribution of sources along the axis, not at least as long as we assume the space to be singly-valued. Blunt ones cannot, but if the meridian is relatively flat at all points, it generally can be done.

The volume of apparent mass can be computed from III (2.1), computing first the normal velocity components and the potential at all points of the surface. The computation is more conveniently performed by the use of the theorem of IV 7.

A special case of this method of obtaining flows produced by the axial motion of solids of revolution is the superposition of a source

and the parallel flow. This does not indeed give a closed surface of stream-lines, but it represents at least one end of such surface extending into infinity, and can be considered as the end of a very long surface of revolution, the other end and its fictitious source or sources being too far away to have any effect on the near end. The pushing ahead of the air by the bow of an airship is indeed very well approximated by one such source.

On account of its simplicity, this flow can conveniently be treated analytically. All fluid generated by the source remains inside the surface and flows to the open side, assuming finally the undisturbed velocity U. The radius of the surface approaches therefore asymptotically a maximum value, say R, which gives the strength of the source $m = \pi R^2 U$. Through all sections separating the source from the open end, the flux is equal to this intensity, and it is zero through all other sections. The velocity being known at all points, this relation allows the computation of the meridian curve. Let the source with the intensity $\pi R^2 U$ be situated at the origin of a system of polar coordinates, and assume $r = f(\theta)$ to represent the meridian. The flux through the spherical cross section $r = constant$ is made up of two parts, that produced by

the source
$$\pi R^2 U \frac{1}{2} (1 - \cos\theta) \tag{5.1}$$

and the flux of the constant velocity U,

$$\pi U (r \sin\theta)^2 \tag{5.2}$$

Their sum is equal to the strength $\pi R^2 U$ of the source, from which we

obtain
$$R^2 = \frac{R^2 (1 - \cos\theta)}{2} + r^2 \sin^2\theta$$

giving
$$r = \frac{R}{2 \sin \frac{\theta}{2}} \tag{5.3}$$

The kinetic energy by III (2.1) is

$$T = \frac{\varrho}{2} \int_0^\pi U R^2 \pi \sin\theta \frac{U R \sin\theta\, 2}{2} d\theta \tag{5.4}$$

The integrand is proportional to

$$\sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

and (5.4) gives
$$T = \frac{\varrho}{2} \cdot \frac{1}{2} \pi R^3 U^2 \frac{1}{3} \sin^3 \frac{\theta}{2} \Big]_0^\pi$$

Hence
$$T = \frac{\varrho U^2}{2} \cdot \frac{1}{6} \pi R^3 \tag{5.5}$$

Each end of the shape discussed is therefore seen to possess a volume of apparent mass equal to one eighth of the volume of the sphere with the maximum diameter, both ends together one fourth of this sphere, Fig. 11.

The fluid passing along the shape in question starts out with the velocity U and finally reaches again the same velocity. It is therefore seen to conserve its momentum, and accordingly it can be expected that

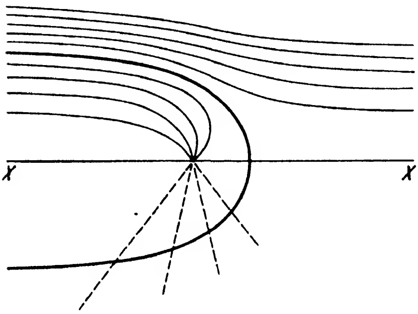

Fig. 11.

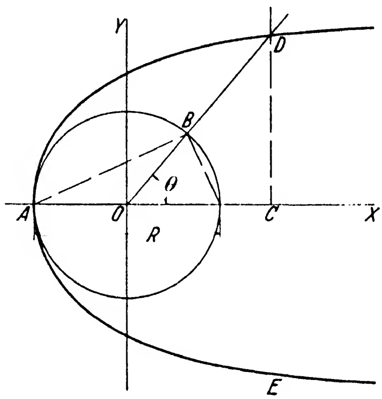

Fig. 12.

the resultant fluid force is equivalent to a pressure equal to that at large distance from the source. This relation can be confirmed by an integration similar to the computation of the resultant force in IV 5.

From (5.3) follows $\bar{y} = r \sin \theta = R \cos \frac{\theta}{2}$

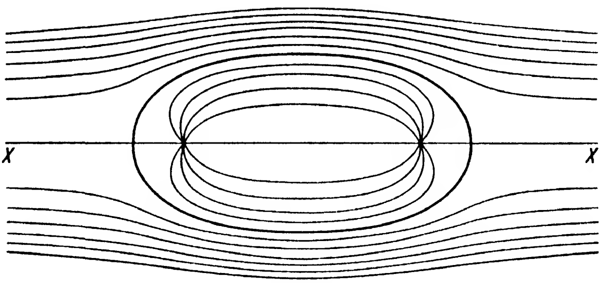

Fig. 13.

This can be used for a convenient construction of the surface equivalent to one source. In Fig. 12, O is the source about which as a center a circle of diameter R is drawn. Then for any radius OB, $COB = \theta$, $CAB = \theta/2$ and $AB = R \cos \theta/2$. We therefore extend OB to a point D where $CD = AB$ and this determines a point on the meridian curve. Other points are similarly found.

Fig. 13 shows the combination of two sources of opposite intensity with the parallel flow. This gives a closed surface.

6. Elongated Surfaces of Revolution. If, in the preceding section, the parallel flow greatly exceeds the sources and sinks in relative strength, the surface shrinks to an elongated surface of revolution with all diameters small when compared with its length. The velocity along the axis approaches the velocity of motion U, and hence the flux through any cross section S approaches US. Since this flux is equal to the sum of all intensities of the sources at either side of the cross section, it becomes now possible to write down a simple formula for the source distribution along the axis

$$m(x) = \frac{d S}{d x} \tag{6.1}$$

where the source intensity m and the cross section S are conceived as functions of the axial coordinate x. The additional apparent mass approaches zero, since the velocities produced are very small when compared with U. The computation gives the same result, for, the static moment of all intensities

$$\int \left(\frac{d S}{d x} x \right) d x \quad \text{is equal to} \quad \int S \, d x \tag{6.2}$$

which is the volume of the elongated surface, leaving nothing for the volume of additional mass. See III 2.

The tangential velocity at any point of the surface depends directly on the distance of the point from the axis. It is therefore not possible, as in the corresponding plane two-dimensional case, to write down simple integrals giving this velocity, containing the source distribution only and the abscissa x. The velocity and the potential can be computed to a higher order of exactness only for special cases which lend themselves to exact mathematical treatment. Section 5 gives such a simple case and is therefore of importance as a basis for the estimate of more complicated shapes. Another case will be discussed in VII 5 and the following ones.

CHAPTER VI

LATERAL MOTION OF SURFACES OF REVOLUTION

1. Doublets. The motion of a perfect fluid produced by a surface of revolution moving at right angles to its axis can be treated along lines similar to those for the axial motion.

The sphere represents both cases at the same time, and its study leads to the general solution. Let x be the axis of a system of semipolar coordinates, x, $\bar{y}$, and ω, where $\bar{y}$ is again the distance from the axis and ω the axial angle. The line $x = 0$, $\omega = 0$, at right angles to the axis, is taken as parallel to the motion. We consider the variation of the potential and of the velocity components along a circle at right angles to the axis. Such circle will be called a parallel circle. In IV (5.5) it was shown that the potential of the doublet with the axis $\perp$ to x can be written in the form $F \cos \omega$, where F is a function of x and $\bar{y}$ only and hence is constant at all points of a parallel circle.

We divide all velocities into two components, one parallel to the plane $\omega = const.$ and one at a right angle to such plane. The latter has the magnitude $\partial \varphi / \bar{y}\, \partial \omega$, and hence is the product of a function of x and $\bar{y}$ only, multiplied by $\sin \omega$.

The former can be divided again into two components u and $\bar{v}$ parallel to the x and $\bar{y}$ direction. Hence either of these is equal to the product of a function of x and $\bar{y}$ multiplied by $\cos \omega$.

It results therefore that in all planes passing through the x axis, the system of curves made up of the components u and $\bar{v}$ of the velocity in any such plane, will be the same shape in all such planes. These curves are not stream-lines since the latter do not lie in these planes. They are, however, made up of components of stream-lines. Such lines are of different density in different planes since all velocities are merely proportional but not equal.

It follows further that if in such a meridian plane, there is to be found a closed curve made up of u and $\bar{v}$ components as above, and across which there is no flux, then the same curve in all the other meridian planes will likewise have the flux zero. This is seen to follow from the fact that for points on two such curves lying on the same parallel circle, the values of x and y are the same and throughout each plane, ω is constant. Since then, the entire system of velocity components u and $\bar{v}$ are proportional to a function of ω, the two systems of velocity components will be similar throughout the two planes and if they are such as to determine a flux zero for one, the same condition will hold for the other.

In the present special case of one doublet combined with the constant parallel flow, these arguments prove that the surface is a sphere, since there exists in this special case symmetry at right angles to the x axis in addition to the axial symmetry parallel to the x axis.

The general argument holds for the doublet and for the parallel flow separately and in conjunction, and we attempt now to generalize the arrangement of the sources without interfering with the conditions giving rise to a surface of revolution as a solid contour. This can be done by choosing an arbitrary distribution of double sources along the x axis, all parallel, say to the y axis, and at right angles to the axis of symmetry of the anticipated surface. Let M denote the strength of the doublets per unit length, where M is now a function of x. The potential of all these doublets, continuously distributed, is then

$$\varphi = \cos \omega \int \frac{M\, \bar{y}\, dx'}{4\pi\, [\bar{y}^2 + (x - x')^2]^{3/2}} \tag{1.1}$$

In this integral, x and $\bar{y}$ are constant during the integration, being the coordinates of the point, for which the potential is computed. x' is variable and denotes the point on the axis at which the strength of the doublet is M. The integral is to be extended over all doublets. If there

are in addition concentrated doublets with the strength M_1, M_2, etc., situated at the points x_1, x_2, etc., there are in addition the terms

$$\frac{M_1 \cos \omega}{4\pi} [\bar{y}^2 + (x - x_1)^2]^{-3/2} \text{ etc.}$$

The potential of the parallel flow was seen to be $\bar{y}\, U \cos \omega$. We see therefore that the superposition of all such parallel doublets, concentrated or continuously distributed along the axis, together with the parallel constant velocity flow, gives a potential which is a product of $\cos \omega$ and a function of x and $\bar{y}$ only. The results derived above stand therefore for any such superposed flows. Such a condition may represent the motion of a solid surface of revolution, moving sidewise, and does so if it represents the motion of a solid at all.

2. Equation of Continuity in Semi-Polar Coordinates. As in the case of axial motion, each distribution of doublets represents the motion caused by a surface of revolution, and, if the intensities are suitably chosen, the motion of a given surface of revolution. This does not always hold conversely. Thus the motion of a surface of revolution very short in axial direction relative to its diameter cannot be represented by a combination of doublets along the axis. This type of motion does, however, find many applications in the problems of aeronautics and we now proceed to establish the differential equation applicable to all lateral motions of surfaces of revolution.

We first compute the flux through the rectangular space element $\delta \bar{y}$, $\bar{y}\delta\omega$, δx. The flux through each single face is

$$\frac{1}{\bar{y}}\frac{\partial \varphi}{\partial \omega}\, \delta \bar{y} \delta x, \qquad \frac{\partial \varphi}{\partial \bar{y}}\, \delta x \bar{y} \delta \omega, \qquad \frac{\partial \varphi}{\partial x}\, \delta \bar{y}\, \bar{y} \delta \omega$$

and the flux through the three pairs of faces

$$\delta x \delta \bar{y} \bar{y} \delta \omega \left[\frac{1}{\bar{y}} \frac{\partial}{\partial \omega} \left(\frac{1}{\bar{y}} \frac{\partial \varphi}{\partial \omega} \right) + \frac{1}{\bar{y}} \frac{\partial}{\partial \bar{y}} \left(\bar{y} \frac{\partial \varphi}{\partial \bar{y}} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) \right]$$

Hence the equation of continuity becomes

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial \bar{y}^2} + \frac{1}{\bar{y}} \frac{\partial \varphi}{\partial \bar{y}} + \frac{1}{\bar{y}^2} \frac{\partial^2 \varphi}{\partial \omega^2} = 0 \tag{2.1}$$

If there is axial symmetry, the last term becomes zero, and we obtain again V (2.2).

In keeping with the preceding discussion we now postulate φ to be a product of a function F of x and $\bar{y}$ only, and of $\cos \omega$.

$$\varphi = \cos \omega\, F(x, \bar{y}) \tag{2.2}$$

Substituting this into (2.1) gives

$$\cos \omega \left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \bar{y}^2} + \frac{1}{\bar{y}} \frac{\partial F}{\partial \bar{y}} - \frac{F}{\bar{y}^2} \right] = 0$$

The entire equation can be divided by $\cos \omega$, and we obtain at last the equation for F,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \bar{y}^2} + \frac{1}{\bar{y}} \frac{\partial F}{\partial \bar{y}} - \frac{F}{\bar{y}^2} = 0 \tag{2.3}$$

which is a partial differential equation with the two independent variables x and $\bar{y}$.

The first three terms of (2.3) are the differential equation, V (2.2) for the flow with axial symmetry. Suppose a potential G has been found complying with V (2.2). Differentiate this equation in G with respect to $\bar{y}$. It results

$$\frac{\partial^3 G}{\partial x^2 \partial \bar{y}} + \frac{\partial^3 G}{\partial \bar{y}^3} + \frac{1}{\bar{y}} \frac{\partial^2 G}{\partial \bar{y}^2} - \frac{1}{\bar{y}^2} \frac{\partial G}{\partial \bar{y}} = 0 \tag{2.4}$$

which shows that $\partial G/\partial \bar{y}$ does not comply with V (2.2) and that the function $\partial G/\partial \bar{y}$ is not symmetric about the axis even if G is. A comparison of (2.3) and (2.4) shows, however, that $\partial G/\partial \bar{y}$ is a solution of the former equation.

3. General Relations. It follows that the velocity component at right angles to the axis of an axially symmetric flow is the potential of a flow of the kind under discussion. The distribution of doublets all parallel and at right angles to the axis is a special case of this more general relation. The doublets result from the differentiation of the analogous distribution of single sources, and have as potential the components of the single sources.

For all these flows, the component stream curves in all meridian planes are alike in shape and proportional in density to the cosine of their axial angle. The peripheral components at right angles to these planes are proportional to the sine of the angle of the plane. These latter velocity components constitute the entire velocity in the plane of symmetry of the flow, $\omega = 90^0$. There, the cross velocity V, at 90^0 to the plane, is directly given by the function F. The meridian flow is the gradient of the cross flow. Hence, the impact point and the point of maximum cross velocity are on the same parallel circle. Either of the two flows can easily be obtained from the other one by differentiation or integration, including the values at the solid surface.

The meridian velocity and the peripheral velocity components being at right angles to each other, the square of the total velocity is equal to the sum of the squares of these two velocity components.

The volume of apparent mass of the solid including its own volume can again be computed by summing up all intensities of the doublets. This gives the apparent mass

$$\frac{\varrho}{U} \int M \, d\, x$$

The method of doublets, as was the case with single sources, is applicable for building up contours to given flows, but not the reverse. It cannot be used for obtaining the flow to a given contour. The combination of the two methods, single sources and doublets fails therefore, to give the flow produced by a given surface of revolution moving in any direction. This would require the knowledge of both, the axial flow and the lateral flow for one and the same surface, since it is the

superposition of these two flows, with suitable intensity, which gives the solution for the motion in all possible directions.

The same distribution of doublets in one case parallel to the axis and in the other at right angles to it, corresponds to solids of revolution of similar type in a way. By no means, however, do they necessarily give the same shape, not even with the intensity varied in a way that may suggest itself for obtaining such agreement.

4. Elongated Surfaces of Revolution. The problem of finding the motion of a perfect fluid produced by a surface of revolution with a given shape has been solved for very elongated surfaces, the maximum diameter of which is small when compared with their length, and the curvature of the meridian curve small enough at all points to avoid sudden changes of the diameter. This solution constitutes the first step toward a more exact solution, and is already exact enough for many practical applications in connection with airship hulls.

The axial motion of such elongated surfaces has been discussed in V 6. The results obtained there are somewhat meager although sufficient for practical application. The results furnished by the application of a similar method to the lateral motion are more complete.

Again we neglect at first the axial velocity components. The flow is then approximated by a motion with all stream-lines in planes at right angles to the axis. These are the two-dimensional flows around the cross section; around the circular cross section in this special case of surfaces of revolution.

The potential of such flow around a circular cylinder is

$$\varphi = \frac{\cos \omega}{y} \cdot \frac{V S}{\pi} \tag{4.1}$$

giving the additional apparent volume equal to the volume of the cylinder itself [1].

The double sources are arranged along the axis of the elongated surface of revolution with an intensity proportional to the cross section of the solid. This follows directly from the assumption of the plane flow and gives the function F

$$F = \frac{S}{\pi\, y}\, V \quad \text{[see (2.2)]} \tag{4.2}$$

and from it the velocity, and in particular, the meridian velocity, which within the same order of approximation as the entire development, is identical with the axial component

$$-\frac{\partial \varphi}{\partial x} = -\frac{\partial F}{\partial x} \cos \omega = -\frac{\partial S}{\partial x}\, \frac{V}{\pi\, y} \cos \omega \tag{4.3}$$

The volume of additional apparent mass of the surface of revolution approaches its own volume as the solid becomes more and more elongated.

[1] See Division B VII 4.

The velocity along the meridian becomes proportional to the rate of change of the cross section along the axis. The velocity across the meridian becomes constant along each meridian. Its maximum becomes equal to V relative to the fluid at large distance, and equal to $2V$ relative to the solid.

We combine now an axial motion of velocity U with the lateral motion V, assuming moreover V to be small when compared with U. This is equivalent to a motion of the solid with a small angle of attack between the axis and the direction of the motion.

The fluid exerts then a resultant moment on the solid, the magnitude of which is computed from III (4.5). The volume of additional apparent mass in the axial direction with great elongation is small or negligible and, as we have just seen, that in the lateral direction is sensibly equal to that of the body itself. The moment with α small will then be

$$M = \frac{\varrho}{2} U^2 \,(\text{Vol.})\, \sin 2\alpha \sim 2\, U^2 \frac{\varrho}{2} \sin \alpha$$

We determine further the distribution of the lateral forces along the axis, giving rise to this resultant moment. The element of lateral momentum $d\mu$, for a layer dx in thickness in a direction at right angles to the axis will correspond to its two-dimensional flow value and is therefore

$$d\mu = \varrho\, S\, dx\, U \sin \alpha \tag{4.4}$$

The momentum per unit thickness along the length is then

$$\frac{d\mu}{dx} = \varrho\, S\, U \sin \alpha$$

The time rate of change of this element of lateral momentum is

$$\frac{\partial}{\partial t}\frac{d\mu}{dx} = \varrho \frac{\partial S}{\partial x}\frac{\partial x}{\partial t} U \sin \alpha = \varrho \frac{\partial S}{\partial x}\, U \sin \alpha\, U \cos \alpha \tag{4.5}$$

The lateral forces are distributed like the sources representing the axial flow, and the forces are in keeping with IV 9.

The same result follows from the examination of the pressure distribution. The velocity at points on the surface is made up (1) of the velocity U of the axial motion (2) of the velocity produced by the axial motion, (3) by the lateral motion and its peripheral components, and (4) by the axial components (4.3) produced by the lateral motion. The square of the velocity contains the constant term U^2, giving rise to a constant pressure not interesting at present. The term of next magnitude is the product of ϱU by the meridian velocity component (4.3).

$$p = \varrho\, U \frac{\partial S}{\partial x} \frac{V}{\pi y} \cos \omega \tag{4.6}$$

This pressure distribution gives the distribution of lateral forces (4.5). For each cross section, the pressure is proportional to $\cos \omega$ and hence to the distance of the point in question from the diameter of the cross section at right angles to the motion.

The squares of the remaining velocity terms and their mutual products are small when compared with this term, except near the ends, in case the ends are round.

The rotation of an elongated surface surrounded by a perfect fluid around an axis intersecting with the axis of symmetry of the surface can be treated in a similar way. Each portion of the solid moves laterally, but now with a velocity different at different points of the axis, and proportional to its distance from the center of motion. Each portion of the surface can be assumed to experience the same fluid reaction as if the whole solid were moving with the velocity of that portion. The additional apparent moment of inertia of the solid is therefore equal to its own inertia, provided the density of the solid is equal to the density of the fluid.

Since the portions near the ends of the elongated surface of revolution contribute comparatively more to the moment of inertia than to the apparent mass in case of lateral motion, and since the approximation is less perfect near the ends, the motion deviating there more distinctly from the plane flow, it can be expected that the discrepancy between the approximation and the exact mathematical method is larger for the moment of inertia than for the apparent mass.

The two expressions should be considered as upper limits rather than as best approximations. The ends allow a part of the fluid to flow sideways and thus to avoid flowing around in planes at right angles to the axis, thereby diminishing the apparent mass and momentum. The expressions are exact only with a really infinite elongation.

5. Equation of Continuity in Polar Coordinates. This equation is obtained from V (4.2) adding to the flux into the rectangular space element that through the faces passing through the axis. This flux is

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \varphi}{r \sin \theta\, \partial \omega} \cdot r\, \delta\, \theta\, \delta r\right) \delta \omega \tag{5.1}$$

The addition of the fluxes through all three pairs of surfaces gives then, after dividing by the volume of the element,

$$\sin \theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{\sin \theta} \frac{d^2 \varphi}{d \omega^2} = 0 \tag{5.2}$$

This is the general equation of continuity in polar coordinates.

6. Tesseral Spherical Harmonics. Transverse motions of surfaces of revolution can also be treated by the use of spherical harmonics. Compare V **3**. We shall not here make any such direct application, but shall need a few of these functions in VII **9** and VII **10** and for this reason insert a few remarks on spherical harmonics not axially symmetrical—the so-called tesseral spherical harmonics.

They are again ordinary algebraic functions of the space coordinates, and we limit ourselves again to integer exponents as in V (3.1). We here,

however, assume the harmonics to depend in a special way on the axial angle ω. In the case of axial symmetry the surface harmonics S are independent of this angle. In the discussion of the lateral motion of surfaces of revolution, potentials were developed in the form of the product of the sine or cosine of ω by a function of the two other polar coordinates only. We generalize these two cases by assuming the potential φ to be of the form

$$\varphi = \cos(s\,\omega)\, P_n^s(r, \theta) \tag{6.1}$$

so that

$$S = \cos(s\,\omega)\, S_n^s(\theta)$$

where P is a function of r, and θ and S a function of θ only. Either the sine or cosine function may be used indifferently. Such functions can be obtained by a differentiation of $1/r$ a number of times with respect to dx and dy and adding the results obtained after multiplication by suitable constant factors. They correspond to certain simple arrangements of sources and sinks of infinite intensity, close to the origin, but no longer arranged along the axis. The simplest cases are:

$$\left.\begin{aligned} P_0' &= \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3} = -\frac{\sin\theta\cos\omega}{r^2} \\ S_0' &= \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\mu^2} \\ P_1' &= \frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) = \frac{3\,x\,y}{r^5} = \frac{3\cos\theta\sin\theta\cos\omega}{r^3} \\ S_1' &= \mu\sqrt{1-\mu^2} \end{aligned}\right| \tag{6.2}$$

From this way of building up tesseral spherical harmonics follow certain useful relations between these functions, and general equations and series for their computations. They are discussed in the literature of the subject. The above two tesseral harmonics are the only ones we shall have to use; their correctness can easily be verified by substituting them in their differential equation. This equation results from substituting (6.1) into (5.2) and is

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dS_n^s}{d\mu}\right] + \left[n(n+1) - \frac{s^2}{1-\mu^2}\right] S_n^s = 0 \tag{6.3}$$

For $s = 0$ it assumes again the form V (4.4). There are also related tesseral functions corresponding to the Q functions V (4.5). They are solutions of the same differential equation, and we shall have use for

$$Q_1' = \sqrt{\zeta^2-1}\left[\frac{1}{2}\log\frac{\zeta+1}{\zeta-1} - \frac{\zeta}{\zeta^2-1}\right]$$

and

$$\left. Q_2' = \sqrt{\zeta^2-1}\left[\frac{3}{2}\zeta\log\frac{\zeta+1}{\zeta-1} - 3 - \frac{1}{\zeta^2-1}\right] \right\} \tag{6.4}$$

The correctness of these solutions is again verified by substitution into (6.3).

CHAPTER VII

ELLIPSOIDS OF REVOLUTION

1. General. The determination of the potential flow produced by an ellipsoid of revolution moving in a perfect fluid otherwise at rest, is the most important three-dimensional solution applying to aeronautical problems. This shape resembles or approximates the hull of airships, and the knowledge of this flow permits therefore, conclusions valuable in connection with the air forces on airship hulls.

The potential flows produced by such motion are ably and completely treated by Horace Lamb in his Hydrodynamics. The reader should compare Lamb's exposition with the following one. He will find in Lamb's treatise many allusions to the mathematical side of the problem not found here. There are also discussed a few types of flows produced by ellipsoids in the process of changing their shapes.

The author has endeavored here to simplify the treatment and completeness in mathematical procedure is sacrificed to greater clarity. This is the more permissible in view of the existence of the fuller treatise of Lamb. The notations are left the same as with Lamb, whenever possible, in order to make the simultaneous reading of the two treatments more convenient. In the present treatment the different steps and their logical relations are given in greater detail and the results have been commented on in somewhat greater fullness.

The entire subject has been divided into two parts, the treatment of the ellipsoid of revolution, and that of the ellipsoid with three different axes. The ellipsoids of revolution require again a different treatment according to whether they are ovary (egg shaped), that is the axis greater than any diameter, or whether they are planetary, with the axis shorter than the largest diameter. The ovary form is more important for the application to aeronautic flow problems than the planetary form. They are therefore treated first. The planetary ellipsoids contain as a special case the circular disc, the treatment of which throws some light on certain phases of the wing theory. Suitable attention is therefore given to this form.

2. Ovary Semi-Elliptic Coordinates. The ovary ellipsoids of revolution require for the computation of their potential flows the introduction of a special system of coordinates, one which possesses the property of orthogonality, and includes the given elliptical surface as one where one coordinate is constant. One of the three coordinates is the axial angle ω discussed in IV 4. The treatment of the ellipsoid with three unequal axes requires a system of coordinates having no element in common with the polar coordinates. This system may be called elliptical coordinates. The present system stands as it were halfway between the elliptical and the polar coordinates and may therefore be called

semi-elliptic coordinates. For further distinction, they may be called "ovary" semi-elliptic, because the planetary ellipsoid require a system somewhat different from that suited to the ovary form.

The origin of the system of semi-elliptic coordinates coincides with the center of the ellipsoid of revolution, the motion of which we proceed to study. It possesses an axis, coincident with the axis of the ellipsoid which we suppose to coincide with the x axis of Cartesian coordinates. A constant angle ω represents planes through this axis. The other two coordinates are denoted by ζ and μ. The former is constant over ellipsoids of revolution and the latter over hyperboloids of revolution of two sheets confocal with the given ellipsoid.

In order to simplify the treatment we assume the given ellipsoid to have the distance 1 between the center and the foci, rather than k as in Lamb's treatment. The scale can always be changed so as to obtain this, and it saves one symbol. We start from the semipolar coordinates x the distance from the plane through the center at right angles to the axis, $\bar{y}$, the distance from the axis, and the axial angle, and we define the new coordinates by means of the following equations

$$\left.\begin{aligned} x &= \mu\,\zeta \\ y &= \sqrt{1-\mu^2}\,\sqrt{\zeta^2-1} \\ \omega &= \omega \end{aligned}\right\} \tag{2.1}$$

Of the new coordinates μ, ζ and ω, the axial angle ω is already known, and we will now study the two others in order to obtain a better grasp of their meaning.

Let μ be constant. We have then $\zeta = x/\mu$

$$\bar{y} = \sqrt{1-\mu^2}\,\sqrt{\frac{x^2}{\mu^2}-1}$$

and hence
$$\frac{x^2}{\mu^2} - \frac{\bar{y}^2}{1-\mu^2} = 1$$

This is the equation of an hyperbola with the foci at $x = \pm 1$, $\bar{y} = 0$ and the semi axes μ and $\sqrt{1-\mu^2}$. From (2.1) it appears that μ has always to be smaller than 1 as otherwise there would occur imaginary values of the coordinates. Such a system of ellipses and hyperbolae is shown in Fig. 14. The constant value of μ for all points of each hyperbola is equal to the distance between where it crosses x and the origin. It follows that in the interval between $x = -1$ and $x = +1$, $x = \mu$ along the axis. It follows further from (2.1) that at these points on the axis $\zeta = 1$.

At a large distance from the origin the hyperbolas approach their asymptotes and hence the shape of straight lines passing through the origin.

$$\frac{y^2}{x^2} = \frac{1-\mu^2}{\mu^2} - \frac{1-\mu^2}{x^2} \tag{2.2}$$

As $\bar{y}$ and x become very large, the last term on the right vanishes, and we obtain

$$\frac{\bar{y}}{x} = \sqrt{\frac{1-\mu^2}{\mu^2}}$$

$\bar{y}/x$ is, however, the tangent of the apex angle θ of the polar coordinates. Equation (2.2) shows therefore that at a large distance from the origin,

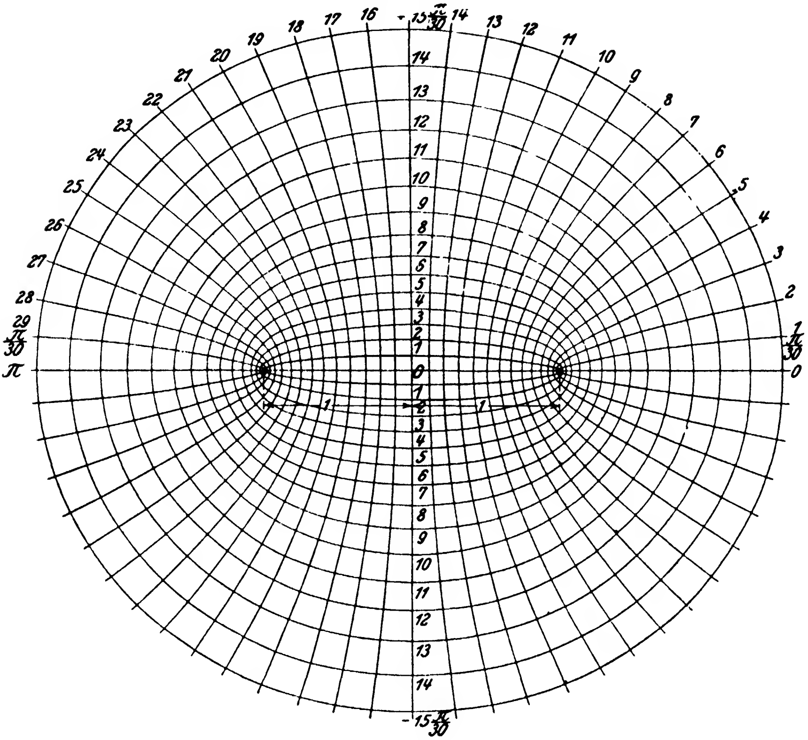

Fig. 14.
(From PRÁŠIL, Technische Hydrodynamik, 2. Aufl., 1926.)

μ approaches the cosine of the apex angle θ, and can therefore be said to be equivalent to, though not equal to the cosine of the apex angle of polar coordinates.

Let next ζ be constant, and μ variable. This gives the equation

$$\mu = \frac{x}{\zeta} \qquad \bar{y} = \sqrt{\zeta^2 - 1}\,\sqrt{1 - \frac{x^2}{\zeta^2}}$$

$$\frac{\bar{y}^2}{\zeta^2 - 1} + \frac{x^2}{\zeta^2} = 1 \tag{2.3}$$

This is the equation of confocal ellipses with the foci at $x = \pm 1$, $\bar{y} = 0$. The semi-axes are ζ and $\sqrt{\zeta^2 - 1}$. Hence, on the x axis, at points outside of the interval between $x = \pm 1$, x agrees with ζ and $\mu = 1$. Equation (2.1) shows ζ to vary in magnitude between 1 and ∞. From

zero to 1, on x for the hyperbolas, x is equal to μ and ζ is 1: beyond, for the ellipses, x is equal to ζ and μ is 1, and equally on the negative side of the axis. The x axis is therefore seen to contain all occurring values of μ and ζ, and indeed intersects with all ellipses and hyperbolas of the system.

When ζ is large, $\zeta^2 - 1$ approaches ζ^2. Equation (2.3) can then be written
$$\bar{y}^2 + x^2 = \zeta^2$$
Hence at large distance from the origin, the ellipsoids of constant ζ approach spheres with ζ as radius. The coordinates ζ, μ, and ω thus correspond to r, $\cos\theta$ and ω of the system of polar coordinates.

3. Equation of Continuity in Ovary Semi-Elliptic Coordinates. The system of coordinates being thus defined and described, we proceed to establish the equation of continuity in terms of the new coordinates. This can be done by means of a mathematical transformation, starting with Laplace's equation II (3.1), and applying the transformation equations (2.1). It is, however, easier to compute the flux through the walls of a space element directly, which element is bounded by three pairs of surfaces of constant semi-elliptical coordinates, each pair corresponding to pairs of coordinates differing by differentials only.

This space element has square corners, the coordinates being orthogonal. Its volume is therefore equal to the product of the three sides. These sides are, however, not equal to $d\mu$, $d\zeta$, and $d\omega$, but are only proportional to these differentials. Their lengths may be denoted by ds_μ, ds_ζ, and ds_ω, respectively. The length of these line elements is computed by differentiating x, y, and z expressed as functions of μ, ζ, and ω with respect to them and by then putting
$$ds_\mu = d\mu \sqrt{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2}$$
The values for x, y, and z are obtained from (2.1) in conjunction with IV (4.1). We thus find,
$$\left.\begin{aligned} ds_\mu &= d\mu \sqrt{\frac{\zeta^2-\mu^2}{1-\mu^2}} \\ ds_\zeta &= d\zeta \sqrt{\frac{\zeta^2-\mu^2}{\zeta^2-1}} \\ ds_\omega &= d\omega \sqrt{(1-\mu^2)(\zeta^2-1)} \end{aligned}\right| \qquad (3.1)$$
This computation may be written down in full for ds_μ
$$x = \zeta\mu, \qquad \frac{\partial x}{\partial \mu} = \zeta$$
$$y = \left(\sqrt{\zeta^2-1}\sqrt{1-\mu^2}\right)\cos\omega$$
$$\frac{\partial y}{\partial \mu} = \frac{-\sqrt{\zeta^2-1}}{\sqrt{1-\mu^2}} \cdot \mu\cos\omega$$

$$z = \left(\sqrt{\zeta^2 - 1}\sqrt{1 - \mu^2}\right) \sin\omega$$

$$\frac{\partial z}{\partial \mu} = \frac{-\sqrt{\zeta^2 - 1}}{\sqrt{1 - \mu^2}}\,\mu \sin\omega$$

Hence $$d s_\mu = d\mu \sqrt{\zeta^2 + \mu^2 \frac{\zeta^2 - 1}{1 - \mu^2}} = d\mu \sqrt{\frac{\zeta^2 - \mu^2}{1 - \mu^2}}$$

The other two line elements are found in a corresponding way.

The velocity in any direction is not equal to the rate of change of the potential with respect to the semi-elliptical coordinates, but to the rate of change with respect to the length of their line elements. This follows from the original definition of the potential, φ as the integral of the scalar product of the velocity and the length of the line element, not the product of the velocity and any coordinate which may happen to be used to express a position on the line. The velocity component in the direction $\zeta = const.$, $\omega = const.$ is therefore $-\partial\varphi/\partial s_\mu$ and similarly in the two other directions. The velocity $-\partial\varphi/\partial s_\mu$ is thus the component at right angles to the surface $\mu = const.$ of the volume element described above. The flux through the pair of these two elements is therefore $$\frac{\partial}{\partial \mu}\left(\frac{\partial \varphi}{\partial s_\mu}\, d s_\zeta\, d s_\omega\right)\delta\mu$$
the product of the last two factors in the bracket being the area of the surface element.

The flux through all three pairs of surface elements is therefore

$$\left.\begin{aligned}\frac{\partial}{\partial \mu}\left(\frac{\partial \varphi}{\partial s_\mu}\, d s_\zeta\, d s_\omega\right)\delta\mu + \frac{\partial}{\partial \zeta}\left(\frac{\partial \varphi}{\partial s_\zeta}\, d s_\mu\, d s_\omega\right)\delta\zeta + \\ + \frac{\partial}{\partial \omega}\left(\frac{\partial \varphi}{\partial s_\omega}\, d s_\mu\, d s_\zeta\right)\delta\omega = 0\end{aligned}\right| \tag{3.2}$$

This flux has to be zero as the condition of continuity.

We substitute the values of the line elements (3.1) into (3.2), obtaining at last the equation of continuity in the form

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \varphi}{\partial \mu}\right] + \frac{\partial}{\partial \zeta}\left[(\zeta^2 - 1)\frac{\partial \varphi}{\partial \zeta}\right] + \frac{\zeta^2 - \mu^2}{(1-\mu^2)(\zeta^2 - 1)}\frac{\partial^2 \varphi}{\partial \omega^2} = 0 \tag{3.3}$$

or transformed

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \varphi}{\partial \mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2 \varphi}{\partial \omega^2} = \frac{\partial}{\partial \zeta}\left[(1 - \zeta^2)\frac{\partial \varphi}{\partial \zeta}\right] + \frac{1}{1-\zeta^2}\frac{\partial^2 \varphi}{\partial \omega^2} \tag{3.4}$$

This is the desired equation.

4. Method of Obtaining Solutions. Equation (3.4) possesses solutions which are the product of three functions, each dependent on one semi-elliptic coordinate only, thus

$$\varphi = F_1(\mu)\cdot F_2(\zeta)\cdot F_3(\omega)$$

It will appear that the solutions which we shall need can be built up from these solutions.

These solutions have the following form:

Either
$$\varphi = \cos(s\,\omega)\, P_n^s(\mu)\, P_n^s(\zeta) \tag{4.1}$$
or
$$\varphi = \cos(s\,\omega)\, P_n^s(\mu)\, Q_n^s(\zeta) \tag{4.2}$$
where in both cases the cosine may be replaced by the sine. The symbols s and n denote any constants, but for the applications we have to make they are integers. The functions P_n^s and Q_n^s are the same as those discussed in V **3** and V **4**.

The correctness of the solutions (4.1) and (4.2) can be verified by substituting them into (3.4). The verification is very simple. It would be much more difficult to explain why the combination (4.1) or (4.2) could be expected to furnish solutions of (3.4).

If, in particular, we have a problem with axial symmetry, s becomes zero. The solutions (4.1) and (4.2) can then be written
$$\varphi = P_n^s(\mu)\, P_n^s(\zeta) \tag{4.3}$$
$$\varphi = P_n^s(\mu)\, Q_n^s(\zeta) \tag{4.4}$$
We shall only use functions with small integral values of s and n, *viz.*, the following ones, discussed in V **3**, V **4**.
$$\left.\begin{aligned} P_0^0(\mu) &= 1 \\ P_1^0(\mu) &= \mu \\ P_2^0(\mu) &= \frac{1}{2}(3\mu^2 - 1) \end{aligned}\right| \tag{4.5}$$
$$\left.\begin{aligned} Q_0^0(\zeta) &= \frac{1}{2}\log\frac{\zeta+1}{\zeta-1} \\ Q_1^0(\zeta) &= \frac{1}{2}\zeta\log\frac{\zeta+1}{\zeta-1} - 1 \\ Q_2^0(\zeta) &= \frac{1}{4}\left[3(\zeta^2-1)\log\frac{\zeta+1}{\zeta-1} - \frac{3}{2}\zeta\right] \end{aligned}\right| \tag{4.6}$$
Of tesseral, unsymmetrical P and Q functions, we shall have occasion to use
$$\begin{aligned} P_1'(\mu) &= \sqrt{1-\mu^2} \\ P_2'(\mu) &= \mu\sqrt{1-\mu^2} \\ Q_1'(\zeta) &= \sqrt{\zeta^2-1}\left[\frac{1}{2}\log\frac{\zeta+1}{\zeta-1} - \frac{\zeta}{\zeta^2-1}\right] \\ Q_2'(\zeta) &= \sqrt{\zeta^2-1}\left[\frac{3}{2}\zeta\log\frac{\zeta+1}{\zeta-1} - 3 - \frac{1}{\zeta^2-1}\right] \end{aligned}$$

5. The Axial Motion of an Ovary Ellipsoid. We assume the ovary ellipsoid to have the semi-focal distance 1. It is supposed to move with the velocity U parallel to its axis in the fluid otherwise at rest. The boundary condition, as discussed in I **11**, has then reference to the velocity component normal to the surface. This surface is given by the condition $\zeta = \zeta_0 = const.$ and hence the normal velocity component is

proportional to, but not equal to $-\partial\varphi/\partial\zeta$. It is equal to $-\partial\varphi/\partial s_\zeta$ and ∂s_ζ is equal in this case to dn. The boundary condition is therefore

$$\frac{\partial \varphi}{\partial s_\zeta} = -U \frac{\partial x}{\partial s_\zeta} \tag{5.1}$$

and hence
$$\frac{\partial \varphi}{\partial \zeta} = -U \frac{\partial x}{\partial \zeta} \tag{5.2}$$

Now, $x = \mu\zeta$ and hence, the final boundary condition is

$$\frac{\partial \varphi}{\partial \zeta} = -U\mu \tag{5.3}$$

The problem is therefore reduced to that of finding a solution of (3.4) in which ω does not occur at all and μ as a direct factor only. The coordinate ζ may occur in any way, since it is constant over the surface, and the solution can therefore always be made to suit the condition (5.3) by the multiplication by a suitable constant multiplier.

The examination of (4.3) and (4.4) in conjunction with (4.5) and (4.6) shows that (4.4) with $n = 1$, $s = 0$, is of the desired kind, and will satisfy the boundary condition (5.3) provided it is multiplied by a suitable factor A. We write therefore

$$\varphi = A\mu\left[\frac{1}{2}\zeta \log \frac{\zeta+1}{\zeta-1} - 1\right] \tag{5.4}$$

If then we have found $\partial\varphi/\partial\zeta$, place it equal to $-\mu U$ from (5.3), put ζ_0 for ζ as the value for the surface and solve for A we shall have

$$A = \frac{U}{\dfrac{\zeta_0}{\zeta_0^2 - 1} - \dfrac{1}{2}\log\dfrac{\zeta_0 + 1}{\zeta_0 - 1}} \tag{5.5}$$

It remains finally to express the quantity ζ_0 by means of the geometrical dimensions of the ellipsoids. In our case, with the half distance of the foci equal to 1, ζ_0 is the large semi-axis, a. The more general case of any focal distance will be covered by the same relations if written in non-dimensional form, and hence it is more general to say that ζ_0 is the ratio of the largert semi-axis a to half the focal distance $\sqrt{a^2 - b^2}$ or

$$\zeta_0 = \frac{a}{\sqrt{a^2 - b^2}} = \frac{1}{e} \tag{5.6}$$

where b is the smaller semi-axis, and e is the eccentricity. The constant A can therefore be written

$$A = \frac{U a}{\dfrac{1}{1 - e^2} - \dfrac{1}{2e}\log\dfrac{1+e}{1-e}} \tag{5.7}$$

where use has been made of the relation $e = 1/a$ valid in our special case.

6. Discussion of the Solution. The potential φ obtained in (5.4) is seen to be an elementary function of μ and ζ, which in turn are elementary functions of x and $\bar{y}$. The relation offers therefore no fundamental difficulties.

The coordinate ζ is constant over the surfaces of all ovary ellipsoids confocal with the given one. Hence the bracket in (5.4) has a constant value over the surface of each confocal ellipsoid. The potential is there proportional to μ and hence to $\mu\,\zeta$ since ζ is constant over the surface. This, however, is x and hence the potential is proportional to x at all points of each confocal ellipsoid. It follows that the axial component is constant.

This holds in particular for the surface of the solid ellipsoid the motion of which is studied, and not only for the flow relative to the fluid at large distance, but also for the flow relative to the ellipsoid itself, since the superposition of a constant velocity parallel to the axis does not interfere with this relation. In the latter case, it is clear that an axial flow over the surface of the ellipsoid must be along meridian lines. It is therefore seen that the velocity of this flow over the surface may be considered as built up of two parts:

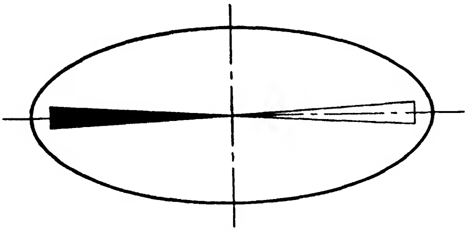

Fig. 15.

(1) A constant velocity along the axis.

(2) A velocity in the lateral direction.

The resultant of these along the normal must be zero, or otherwise, the components of (1) along the normal must be equal and opposite to those of (2). The combination of these facts with a knowledge of the direction of the velocity can be used with advantage for a convenient computation of the velocity, and from it of the pressure by the use of II (4.2).

We inquire next regarding the distribution of fictitious sources along the axis, that would give rise to this flow. To this end we transform the semi-elliptic coordinates in (5.4) into semipolar coordinates under the simplifying condition that $\bar{y}$ is small.

$$x = \mu\,\zeta \sim \mu\ (since\ \zeta = 1)$$

$$\bar{y} = \sqrt{1-\mu^2}\,\sqrt{\zeta^2-1} \sim 0\ (\bar{y}\ \text{small})$$

$$\zeta = \sqrt{1+\frac{\bar{y}^2}{1-x^2}} = 1 + \frac{1}{2}\frac{\bar{y}^2}{1-x^2} + \ldots\ (\bar{y}\ \text{small})$$

Hence (5.4) becomes $\quad \varphi = A\,x\,\frac{1}{2}\,log\,\frac{4(1-x^2)}{\bar{y}^2} + \ldots$

or
$$\varphi = A\,x\left[log\,2 + \frac{1}{2}\,log\,(1-x^2) - log\,\bar{y} + \ldots\right] \tag{6.1}$$

The term $log\ \bar{y}$ in the bracket indicates a uniform distribution of sources along the axis, as shown by comparison with the potential of a two-dimensional source. This term multiplied by x shows therefore that

between the foci, the intensity of the sources is proportional to the distance from the center, Fig. 15.

Outside the foci, the development (6.1) holds no longer. The expression under the log does not vanish there, and hence there are no sources outside the foci.

The distribution of the sources is therefore seen to have almost the same arrangement for any ovary ellipsoid as for one with infinite elongation, except that in the latter case the sources extend along the entire axis, and in the general case they extend between the foci only.

This is in keeping with the general character of the solution. The same solution serves for all confocal ellipsoids, except that the ratio of the constant multiplier A to the velocity U changes. Hence, leaving the sources unchanged, and varying the velocity U from a very large value to one smaller and smaller, swells, the ellipsoid from a very elongated one with its entire axis occupied by sources to one fatter and fatter, the axis becoming longer too, until at last a very large sphere is obtained. The distribution of the sources can therefore be computed from the special case of the very elongated ellipsoid extending between the foci. The stream-lines have, moreover, the same shape for all excentricities. They are shown in Fig. 16.

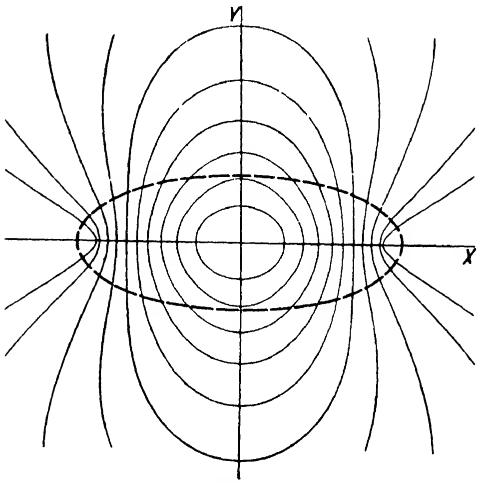

Fig. 16.

7. Apparent Mass. The kinetic energy is computed by III (2.1). The computation is very simple in the present case, because the potential over the surface, where $\zeta = \zeta_0$ a constant, becomes proportional to μ and since $x = \mu\,\zeta_0$ (over the surface) the potential is proportional to x or to the distance of the point on the surface from the plane of the equator. The potential over the surface may therefore be written $\varphi = c x$. Divide the ellipsoid into many narrow tubes parallel to the axis, with the cross section $dy\,dz$. The flux through each surface element cut out by the tube is then $U dy\, dz$. The potential difference between the two intersections of the tube with the surface is cl where l denotes the length of the tube. Hence the kinetic energy is

$$\frac{\varrho}{2}\int\int U\,d\,y\,d\,z\,c\,l = \frac{\varrho\,c\,U}{2}\int\int\int d\,x\,d\,y\,d\,z \tag{7.1}$$

This is equal to the volume of the ellipsoid multiplied by $\varrho\, c\, U/2$. From which follows that the coefficient of apparent volume, k_1 in III 2 is equal to

$$k_1 = \frac{c}{U} = \frac{A}{U}\left[\frac{1}{2} \log \frac{\zeta_0 + 1}{\zeta_0 - 1} - \frac{1}{\zeta_0}\right] \tag{7.2}$$

Substituting $e = 1/\zeta_0$ and introducing the value of A (5.5) gives

$$k_1 = \frac{\gamma}{2 - \gamma} \tag{7 3}$$

where

$$\gamma = 2\left(\frac{1 - e^2}{e^3}\right)\left[\frac{1}{2} \log \frac{1 + e}{1 - e} - e\right] \tag{7.4}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{7.5}$$

where a denotes the half axis and b the largest radius of the ellipsoid.

Table 1 contains k_1 as function of b/a. The ratio of the maximum tangential velocity to the velocity of motion is likewise k_1 in the case with the fluid at rest except for the motion of the ellipsoid. With the ellipsoid at rest and the fluid moving, the maximum tangential velocity is $(1 + k_1)$ times the velocity at a great distance.

TABLE 1.

Inertia Factors of Ellipsoids of Revolution For Axial Motion, Lateral Motion and Rotation.

a/b	k_1	k_2	k'	a/b	k_1	k_2	k'
1.00	0.500	0.500	0.000	6.01	0.045	0.918	0.764
1.50	0.305	0.621	0.094	6.97	0.036	0.933	0.805
2.00	0.209	0.702	0.240	8.01	0.029	0.945	0.840
2.51	0.156	0.763	0.367	9.02	0.024	0.954	0.865
2.99	0.122	0.803	0.465	9.97	0.021	0.960	0.883
3.99	0.082	0.860	0.608	∞	0.000	1.000	1.000
4.99	0.059	0.895	0.701				

8. The Stream Function. In the present case of axial symmetry, there exists a stream function, as discussed in V 1. Its knowledge is of use for the drawing of the stream-lines $\psi = const.$ The stream function can be obtained from the potential by means of V (1.1). These equations obviously hold for any line elements at right angles to each other, since nothing in their derivation refers to the direction of the line elements relative to the axis. Hence we have

$$\frac{\partial \varphi}{\partial s_\zeta} = -\frac{1}{y}\frac{\partial \psi}{\partial s_\mu} \quad \text{and} \quad \frac{\partial \varphi}{\partial s_\mu} = \frac{1}{y}\frac{\partial \psi}{\partial s_\zeta} \tag{8.1}$$

Introducing the magnitude of the line elements

$$\frac{\partial \psi}{\partial \mu} = -(\zeta^2 - 1)\frac{\partial \varphi}{\partial \zeta} \tag{8.2}$$

and from (5.4)

$$\frac{\partial \psi}{\partial \mu} = -(\zeta^2 - 1) A \mu \left[\frac{1}{2} \log \frac{\zeta + 1}{\zeta - 1} - \frac{\zeta}{\zeta^2 - 1}\right]$$

Hence $$\psi = -\frac{A}{2} (\zeta^2 - 1) [\mu^2 - f(\zeta)] \left[\frac{1}{2} \log \frac{\zeta + 1}{\zeta - 1} - \frac{\zeta}{\zeta^2 - 1}\right] \tag{8.3}$$

$f(\zeta)$ is to be determined from the relation

$$\frac{\partial \psi}{\partial \zeta} = (1 - \mu^2) \frac{\partial \varphi}{\partial \mu} = A (1 - \mu^2) \left[\frac{1}{2} \zeta \log \frac{\zeta + 1}{\zeta - 1} - 1\right] \tag{8.4}$$

This equation shows $f(\zeta)$ to be equal to 1. The stream function of the ovary ellipsoid moving axially results therefore

$$\psi = \frac{1}{2} A (1 - \mu^2) (\zeta^2 - 1) \left[\frac{1}{2} \log \frac{\zeta + 1}{\zeta - 1} - \frac{\zeta}{\zeta^2 - 1}\right] \tag{8.5}$$

9. Lateral Motion of the Ovary Ellipsoid. The computation of this flow follows closely the preceding computation, and can therefore be treated briefly.

The boundary condition similar to (5.2) is

$$\frac{\partial \varphi}{\partial \zeta} = -U \frac{\partial y}{\partial \zeta} \tag{9.1}$$

This is satisfied by putting in (4.2) $n = 1$, $s = 1$ giving,

$$\varphi = A \sqrt{1 - \mu^2} \sqrt{\zeta^2 - 1} \left[\frac{1}{2} \log \frac{\zeta + 1}{\zeta - 1} - \frac{\zeta}{\zeta^2 - 1}\right] \cos \omega \tag{9.2}$$

A being given by $$A \left[\frac{1}{2} \log \frac{\zeta_0 + 1}{\zeta_0 - 1} - \frac{\zeta_0^2 - 2}{\zeta_0 (\zeta_0^2 - 1)}\right] = -U \tag{9.3}$$

At the surfaces of all confocal ellipsoids, more especially at the surface of the solid, the potential is seen to be proportional to

$$\sqrt{1 - \mu^2} \sqrt{\zeta^2 - 1} \cos \omega$$

and hence to y, giving rise to the same theorem regarding the velocity distribution over the surface as in the preceding investigation. It is again the component of a constant velocity parallel to the motion. This, together with the relations discussed in VI 3 allows a convenient computation of the velocities and from it of the pressure distribution. The computation of the inertia factor can therefore be performed in the same way as in 7. It results

$$k_2 = \frac{\alpha}{2 - \alpha} \tag{9.4}$$

where $$\alpha = \frac{1}{e^2} - \frac{1 - e^2}{2 e^2} \log \frac{1 + e}{1 - e}$$

Its value is tabulated in Table 1.

We determine at last the distribution of the doublets along the axis. Between the foci

$$x = \mu; \qquad \zeta^2 - 1 = \frac{\bar{y}^2}{1 - x^2}; \qquad \zeta = 1 + \frac{1}{2} \frac{\bar{y}^2}{1 - x^2} + \cdots\cdots$$

and hence $$\varphi = A \bar{y} \cos \omega \left[\frac{1}{2} \log \frac{4 (1 - x^2)}{\bar{y}^2} - \frac{1 - x^2}{\bar{y}^2} + \cdots\cdots\right]$$

The second term in the bracket indicates the doublet, the characteristic term being *cos* $\omega/\bar{y}^2$. This term is associated with a factor containing x in the form $(1 - x^2)$. Outside the foci there is no term *cos* $\omega/\bar{y}^2$.

Hence the distribution of the doublets is again the same as with a very elongated ellipsoid, it is parabolic and proportional to the cross sections of such elongated ellipsoid extended between the foci, not along the entire axis of the ellipsoid. The same flow serves again for ellipsoids of all eccentricities, and the distribution of the doublets could therefore be expected from the special case of the very elongated ellipsoid.

10. Rotation of the Ovary Ellipsoid. If the ovary ellipsoid rotates with the angular velocity Ω about the largest diameter drawn in the y direction, the boundary condition is

$$\frac{\partial \varphi}{\partial \zeta} = -\Omega \left(z \frac{\partial x}{\partial \zeta} - x \frac{\partial z}{\partial \zeta} \right)_{\zeta = \zeta_0} \tag{10.1}$$

This gives

$$\frac{\partial \varphi}{\partial \zeta} = \frac{\Omega}{\sqrt{\zeta_0^2 - 1}} \mu \sqrt{1 - \mu^2} \sin \omega \tag{10.2}$$

The solution is obtained by putting in (4.2) $n = 2$, $s = 1$.
There results the potential

$$\varphi = A \mu \sqrt{1 - \mu^2} \sqrt{\zeta^2 - 1} \sin \omega \left[\frac{3}{2} \zeta \log \frac{\zeta + 1}{\zeta - 1} - 3 - \frac{1}{\zeta^2 - 1} \right] \tag{10.3}$$

A is determined from (10.2) and (10.3), first differentiating (10.3) with respect to ζ and then equating the result with (10.2). The potential over all points of the same confocal ellipsoids is proportional to

$$\mu \sqrt{1 - \mu^2} \sqrt{\zeta^2 - 1}$$

and hence to $x z$. This holds in particular also for all points on the surface of the ellipsoid.

This relation simplifies the computation of the apparent moment of inertia, the particular form of the potential leading to a simple relation of the apparent moment to the moment of inertia of the solid, in the same manner as the potential for the straight motion leads to the apparent volume.

The kinetic energy of the fluid is equal to the integral over the fluid displaced by the surface elements of the rotating solid, multiplied by the value of the potential at each point [see III (2.1)].

$$2T = \varrho \int\int \varphi \frac{\partial \varphi}{\partial n} ds \tag{10.4}$$

Let Ω be the angular velocity about the y axis. Then the fluid displaced per unit time by a surface element rotating in the manner specified is

$$\Omega\, x\, dx\, dy - \Omega\, z\, dz dy$$

The potential for ζ constant or over the surface is, as we have seen, proportional to $x z$ and may be written in the form $\varphi = C x z$ where

C is a constant. The energy corresponding to this potential and flux will therefore be given by

$$2\,T = \varrho\, C\, \Omega \iint [x^2 z\, d x\, d y - x\, z^2\, d z\, d y] \tag{10.5}$$

This surface integral can now be transformed into a space integral throughout the entire ellipsoid, by differentiation of the two terms with respect to dz and dx respectively. This is the same method as used in I 2, in connection with Gauss' theorem.

The resulting space integral is

$$2\,T = \varrho\, C\, \Omega \iiint (x^2 - z^2)\, d x\, d y\, d z \tag{10.6}$$

This integral represents the difference of the moments of inertia of the ellipsoid with respect to the yz plane and the xy plane.

These moments of inertia are easily computed. The moment of inertia of a spherical shell with the radius r and the thickness dr, relative to its center is $4\pi r^4\, dr$ from which the moment of inertia of the solid sphere results as the integral of this expression, giving $\frac{4}{5}\pi r^5$. The moment of inertia of the same sphere relative to a plane through its center is one third of this, or $\frac{4}{15}\pi r^5$.

Stretching now the sphere into an ellipsoid by increasing all coordinates in a certain ratio increases the moment of inertia directly with this ratio if the stretching takes place parallel to the plane of reference, with the third power of the ratio if the stretching is performed at right angles to the plane. Hence the moment of inertia of the ellipsoid relative to the $x\,y$ plane is $\frac{4}{15}\pi a\, b^4$ where a is the half axis along x, the axis of symmetry and b is the half axis $\perp$ to x. Similarly $\frac{4}{15}\pi a^3\, b^2$ will be the moment about the $y\,z$ plane.

It follows that the difference of the moments of inertia, represented by the integral (10.6) is $\frac{4}{15}\pi a\, b^2\,(a^2 - b^2)$. For C we have the value from (10.3)
$$C = \frac{A}{\zeta_0}\left[\frac{3}{2}\,\zeta_0 \log \frac{\zeta_0+1}{\zeta_0-1} - 3 - \frac{1}{\zeta_0^2-1}\right]$$

This leads to the value of k', the ratio of the apparent moment of inertia to the moment of inertia of the volume: $(4/15)\pi a\, b^2\,(a^2 + b^2)$

$$k' = \frac{e^4\,(\alpha - \gamma)}{(2 - e^2)\,[2\,e^2 - (2 - e^2)\,(\alpha - \gamma)]} \tag{10.7}$$

where α and γ have the same meaning as in 7 and 9.

The value of k' is tabulated in Table 1 for several ratios of the axes of the ellipsoid.

11. Modification of the Method for Planetary Ellipsoids. The ovary semi-elliptical coordinates are not entirely unsuitable for the representation and solution of flow problems relating to planetary ellipsoids of revolution—those with the axis smaller than the greatest diameter.

It is merely necessary to employ imaginary coordinates. The general results of the preceding sections hold therefore for planetary ellipsoids.

It is possible and often more convenient to write all equations without the use of the imaginary, by substituting ζ' for ζ such that $\zeta'^2 = -\zeta^2$. The semi-elliptic coordinates are then defined by means of:

$$x = \mu\zeta \qquad \bar{y} = \sqrt{1-\mu^2}\ \sqrt{\zeta^2+1} \tag{11.1}$$

Along the entire x axis $\zeta = x$, varying now from $-\infty$ to $+\infty$ throughout all values. $\zeta = const.$ and $\mu = const.$ are again all confocal ellipses and hyperbolas, with the foci at the points $x = 0$, $y = \pm 1$. ζ is again the semi-axis of its ellipse on the x axis, which is now the smaller axis. The hyperboloids of revolution are now of one sheet.

The line elements have the value

$$\left.\begin{aligned} ds_\mu &= d\mu \frac{\sqrt{\zeta^2+\mu^2}}{\sqrt{1-\mu^2}} \\ ds_\zeta &= d\zeta \frac{\sqrt{\zeta^2+\mu^2}}{\sqrt{\zeta^2+1}} \\ ds_\omega &= d\omega \sqrt{1-\mu^2}\ \sqrt{\zeta^2+1} \end{aligned}\right\} \tag{11.2}$$

and the equation of continuity reads

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\varphi}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2\varphi}{\partial\omega^2} = -\frac{\partial}{\partial\zeta}\left[(\zeta^2+1)\frac{\partial\varphi}{\partial\zeta}\right] + \frac{1}{\zeta^2+1}\frac{\partial^2\varphi}{\partial\omega^2} \tag{11.3}$$

The special solutions required in this case are

$$\varphi = P_n^s,\ q_n^s \tag{11.4}$$

where the P functions are the same as before, but the q functions are different, their differential equation having changed in keeping with (11.1). These functions are

$$\left.\begin{aligned} q_0^0 &= cot^{-1}\zeta \\ q_1^0 &= 1-\zeta\, cot^{-1}\zeta \\ q_2^0 &= \frac{1}{2}(3\zeta^2+1)\, cot^{-1}\zeta - \frac{3}{2}\zeta \\ q_n^s &= \sqrt{(\zeta^2+1)^s}\ \frac{d^s}{\partial\zeta^s} q_n^0 \end{aligned}\right\} \tag{11.5}$$

and hence, since $\dfrac{d\, cot^{-1} x}{dx} = -\dfrac{1}{1+x^2}$

$$\left.\begin{aligned} q_1^1 &= -\sqrt{\zeta^2+1}\left(cot^{-1}\zeta - \frac{\zeta}{\zeta^2+1}\right) \\ q_2^1 &= \sqrt{\zeta^2+1}\left(3\zeta\, cot^{-1}\zeta - 3 + \frac{1}{\zeta^2+1}\right) \end{aligned}\right\} \tag{11.6}$$

These modifications give for the axial motion of the planetary ellipsoid the solution

$$\left.\begin{aligned} \varphi &= A\mu(1-\zeta\, cot^{-1}\zeta) \\ \psi &= \frac{A}{2}(1-\mu^2)(\zeta^2+1)\left[\frac{\zeta}{\zeta^2+1} - cot^{-1}\zeta\right] \end{aligned}\right\} \tag{11.7}$$

with the multiplier $$A = -\frac{U}{\frac{\zeta_0}{(\zeta_0^2+1)} - \cot^{-1}\zeta_0} \qquad (11.8)$$

If the half axis is a, and the largest radius is b the eccentricity will now be

$$e = \frac{\sqrt{b^2 - a^2}}{b}$$

and we have

$$a = \zeta_0$$
$$b = \sqrt{\zeta_0^2 + 1}$$
$$e = \frac{1}{\sqrt{\zeta_0^2+1}}$$

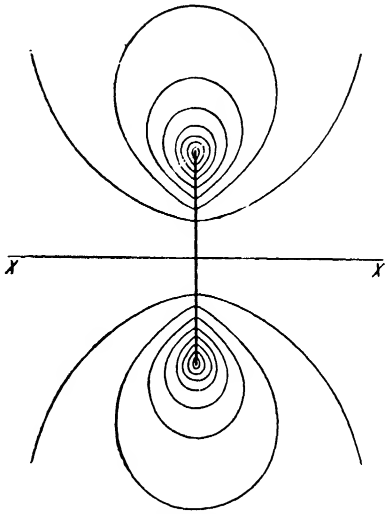

Fig. 17.

This gives A in terms of the eccentricity

$$A = -\frac{Ub}{\sqrt{1-e^2} - \frac{1}{e}\sin^{-1} e}$$

For the circular disc we have $e = 1$ and $\zeta_0 = 0$. The potential becomes then

$$\varphi = \pm A\mu$$
$$= \pm A\sqrt{1 - \frac{y^2}{b^2}}$$

on the two sides of the disc. The normal outside velocity at the points of the disc will be U. This gives the kinetic energy by III (2.1).

$$T = \frac{8}{3} b^3 \varrho \frac{U^2}{2}$$

This corresponds to a volume of apparent mass equal to $2/\pi$ times the volume of the sphere with the same diameter as the disc.

The lines of this flow, which are the same for all planetary ellipsoids with the same focal distance, are shown in Fig. 17.

For the motion of the planetary ellipsoid parallel to a largest diameter we have again in (11.4) $n = 1$, $s = 1$ with the same boundary condition as with the ovary ellipsoid. This gives

$$\varphi = A\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos\omega\left[\frac{\zeta}{\zeta^2+1} - \cot^{-1}\zeta\right]$$

with $$A = -\frac{U}{\frac{\zeta_0^2+2}{\zeta_0(\zeta_0^2+1)} - \cot^{-1}\zeta_0}$$

The disc moving parallel to its face does not give rise to any fluid motion at all, as would be expected. The lateral flow around the planetary ellipsoid has so far not found any practical application for flight problems. There is a remote connection between these flows with certain questions of wing theory, referring to wings with finite span.

We discuss finally the fluid motion produced by a planetary ellipsoid rotating about one of its largest diameters. The solution is again given by (11.4) with $n = 2$ and $s = 1$. This gives

$$\varphi = A \mu \sqrt{1 - \mu^2} \sqrt{\zeta^2 + 1} \sin \omega \left[3 \zeta \cot^{-1} \zeta - 3 + \frac{1}{\zeta^2 + 1} \right]$$

with the surface condition

$$\frac{\partial \varphi}{\partial \zeta} = -\Omega_y \left(z \frac{\partial x}{\partial \zeta} - x \frac{\partial z}{\partial \zeta} \right)$$

For the circular disc ($\zeta_0 = 0$) this gives

$$\frac{3}{2} \pi A = -\Omega_y$$

and using the formula $T = -\frac{\varrho}{2} \int \int \varphi \frac{\partial \varphi}{\partial n} \bar{y}\, d\bar{y}\, d\omega$

$$T = \frac{16}{45} b^5 \frac{\varrho}{2} \Omega^2$$

The apparent moment of inertia of the disc about the axis of rotation is therefore $\frac{2}{3\pi}$ of the moment of inertia $\frac{8}{15} \pi r^5$ of the sphere having the same diameter and the density of the fluid.

12. Most General Motion of Ellipsoids of Revolution. Any motion of an ellipsoid of revolution can be obtained by the superposition of the special motions discussed in the preceding sections. The potential of the superposed motion is then the sum of the single potentials of its components. If the motion is purely rectilinear, each single potential is a linear function of the space coordinates for all points of the surface, and hence their sum is likewise, as this property is not destroyed by the addition. The gradient of this linear potential does not however remain parallel to the motion in general, but its relation to the motion is dyadic. We shall see in the next chapter, that this relation holds also for the most general motion of ellipsoids with three different axes as long as the motion is straight. Since the tangential velocity at the surface is equal to the component of this maximum velocity, it is equal at all points, the tangential plane of which is parallel to the maximum velocity. There is therefore no single point of maximum velocity and minimum pressure, but rather the maximum velocity and minimum pressure are constant along an ellipse all around the ellipsoid.

For very elongated ellipsoids the ratio of the maximum velocity relative to the solid to the velocity of motion is almost 2 for lateral motion and 1 for axial motion. Hence, for small angles of attack, the angle between the axis and the maximum velocity is almost twice the angle of attack of the axis.

CHAPTER VIII

ELLIPSOID WITH THREE UNEQUAL AXES

1. Remarks on Elliptical Coordinates. The motion of a perfect fluid at rest except for the effect of the translation of an ellipsoid with three unequal axes is studied by means of elliptic coordinates. The mathematical treatment of these coordinates is simplified by their symmetry. The equation of definition between the elliptical coordinates λ, μ and ν and the Cartesian coordinates x, y, and z is one and the same for all three of them. This does not mean that the three elliptical coordinates at each point have the same value. They have different values, the equation defining them being of the third degree and possessing three different roots, one for each elliptical coordinate.

This symmetry is not confined to the definition, but extends to all equations occurring. All equations can be arranged in groups of three, resulting from each other by substituting x for y, y for z, z for x, λ for μ, μ for ν, and ν for λ. This symmetry gives to all developments interest and beauty from the mathematical point of view, and is of practical assistance in deducing the desired relations. Such simplification is indeed welcome, for the treatment of elliptical coordinates requires somewhat more mathematical skill than the treatment of the semi-elliptical coordinates, which indeed are a special case of them. The semi-elliptical coordinates are unsymmetric with respect to the Cartesian coordinates, the unequal axes of the ellipsoid playing a part different from that of the two equal axes. They provide however, a solution of the problem by means of elementary explicit functions. In this way we are able to write down equations with one semi-elliptical coordinate on one side and a function of the Cartesian coordinates on the other, with other equations developed from these in the same form. With elliptic coordinates, this is indeed likewise possible, but it is not practical since the expressions become too involved. We must therefore be satisfied with equations of the form $\quad f(x, y, z, \lambda) = 0$

The elliptical coordinates possess two features in common with the semi-elliptic coordinates: on the surface of the solid ellipsoid the motion of which is studied, one elliptic coordinate, which we shall denote by λ, is constant; and further, the surfaces of constant elliptical coordinates are orthogonal—they intersect with each other at right angles.

2. Equation of Continuity in Elliptical Coordinates. The elliptical coordinates λ, μ, and ν are connected with the Cartesian coordinates x, y, and z, by means of the following three equations, which are identical with each other:

$$\left.\begin{aligned} \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 &= 0 \\ \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} - 1 &= 0 \\ \frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} + \frac{z^2}{c^2+\nu} - 1 &= 0 \end{aligned}\right\} \qquad (2.1)$$

Each of these equations defines one coordinate, and we will stipulate that $a > b > c$ and that λ lies between ∞ and $-c^2$, μ between $-c^2$ and $-b^2$, and ν between $-b^2$ and $-a^2$.

The surfaces for $\lambda = const.$ are then ellipsoids,
those for $\mu = const.$ are hyperboloids of one sheet,
and those for $\nu = const.$ are hyperboloids of two sheets

While it is not practical to write λ, μ, and ν explicitly as functions of x, y, and z, it is easy to write down the inverse relations explicitly, that is x, y, and z as functions of λ, μ, and ν. Equations (2.1) are linear in x^2, y^2, and z^2 and hence can be solved for them by well known methods.

It results
$$\left.\begin{aligned} x^2 &= \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 &= \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)} \\ z^2 &= \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} \end{aligned}\right\} \tag{2.2}$$

If we introduce into (2.1) a value θ for λ, which is not equal to one of the three elliptical coordinates corresponding to the values of x, y, and z, the right hand side of the equation will no longer be zero. Its value can either be expressed by the Cartesian coordinates, or by the elliptical coordinates, or indeed by any combination of the six. We proceed to express this value by means of the elliptical coordinates, which is done by substituting θ for λ in (2.1) and x^2, y^2, z^2 from (2.2). It results
$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1=\frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)} \tag{2.3}$$

It can be directly seen from the form of the expression on the right hand side that this gives zero for $\theta = \lambda$, $\theta = \mu$, $\theta = \nu$ and infinity for $\theta = -a^2$, $\theta = -b^2$, $\theta = -c^2$.

In order to obtain the equation of continuity in elliptical coordinates, the first step must be directed toward the expressions for the line elements corresponding to small changes of the elliptical coordinates. We differentiate first (2.2) with respect to λ, μ, and ν and obtain in this way expressions for $\partial x/\partial \lambda$ and the other eight similar combinations, in a mixed form, containing one Cartesian coordinate and one elliptical coordinate.
$$\frac{\partial x}{\partial \lambda}=\frac{x}{2(a^2+\lambda)}\,; \qquad \frac{\partial y}{\partial \lambda}=\frac{y}{2(b^2+\lambda)}\,; \qquad \frac{\partial z}{\partial \lambda}=\frac{z}{2(c^2+\lambda)} \tag{2.4}$$

It must be understood that in itself it is of no importance whether an expression is given in elliptical or Cartesian coordinates, or mixed, since all of these six coordinates are known if any three are given. When we differentiate with respect to one of the six coordinates, we must specify which quantities are to be assumed constant and which not, for when proceeding from one point in space to another, we cannot

keep more than two coordinates constant. We therefore lay down the rule that when differentiating partially with respect to any coordinate, we shall consider constant the two other coordinates belonging to the same system, Cartesian if the differentiation refers to a Cartesian coordinate, and elliptical if to an elliptical one. Thus, $dx/\partial\lambda$ means the rate of change of x with respect to λ at the point considered when proceeding along the line $\mu = const.$, $\nu = const.$

The derivative $\partial\lambda/\partial x$, however, is by no means the inverse of $\partial x/\partial\lambda$, because it is not the inverse ratio of the same quantities but of different ones. It is the rate of change of λ with respect to x when proceeding from the same point as before with y and z constant. The relation of these two derivatives must now be considered.

We obtain mixed expressions for the line elements by using (2.4) substituting them into

$$ds_\lambda^2 = \left[\left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2\right] d\lambda^2$$

obtaining thus

$$\left(\frac{ds_\lambda}{d\lambda}\right)^2 = \frac{1}{4}\left[\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right]$$

This equation still contains both kinds of coordinates. For further reduction, we put, for convenience

$$\frac{1}{h_1} = \frac{ds_\lambda}{d\lambda}\ ;\ \frac{1}{h_2} = \frac{ds_\mu}{d\mu}\ ;\ \frac{1}{h_3} = \frac{ds_\nu}{d\nu} \tag{2.5}$$

and substitute in this equation for x^2, y^2, and z^2 from (2.2). We thus have:

$$\left.\begin{aligned} h_1^2 &= 4\,\frac{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)} \\ h_2^2 &= 4\,\frac{(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\mu-\nu)(\mu-\lambda)} \\ h_3^2 &= 4\,\frac{(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\lambda)(\nu-\mu)} \end{aligned}\right\} \tag{2.6}$$

where the two latter are written down from symmetry.

We are now ready to express the relation between the two derivatives $\partial\lambda/\partial x$ and $\partial x/\partial\lambda$. The symbol ds_λ signifies the shortest distance between two surfaces $\lambda \pm d\lambda/2 = const.$, which surfaces are separated at the point considered by elements dx parallel to the x axis, and which are not the shortest connections. Therefore, ds_λ is always smaller or at best equal to dx, but never larger, and their ratio ds_λ/dx is the cosine of the angle between the normal to $\lambda = const.$ and the x axis. Compare with this, dx/ds_λ where the differentiation is carried on by proceeding at a right angle to $\lambda = const.$ This is the x component of the distance travelled when increasing λ by $d\lambda$ travelling at a right angle to the elliptical surface $\lambda = const.$, and is therefore again equal to the cosine

of the angle mentioned before. Hence with (2.5) we have the equation appearing at first glance incorrect:

$$\frac{\partial s_\lambda}{\partial x} = \frac{\partial x}{\partial s_\lambda}\,; \qquad \frac{1}{h_1}\frac{\partial \lambda}{\partial x} = h_1 \frac{\partial x}{\partial \lambda}\,; \qquad \frac{\partial \lambda}{\partial x} = h_1^2 \frac{\partial x}{\partial \lambda} \tag{2.7}$$

which together with (2.4) and (2.6) gives

$$\frac{\partial \lambda}{\partial x} = 2\,x\,\frac{(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)} \tag{2.8}$$

a formula for which use will be found at a later point.

Having by (2.6) obtained the means for expressing the length of all line elements by elliptical coordinates only, we proceed to express the divergence of a potential φ in terms of the elliptic coordinates. We do so by computing the flux through the faces of a small rectangular space element with the elliptical coordinates constant on its faces. The velocity component parallel to ds_λ, that is normal to $\lambda = const.$, is $h_1\,(\partial\varphi/\partial\lambda)$. Hence the flux through the face with the sides ds_μ, ds_ν, (at right angles to ds_λ) is $\quad h_1 \frac{\partial \varphi}{\partial \lambda}\, d s_\mu\, d s_\nu = \frac{h_1}{h_2 h_3}\frac{\partial \varphi}{\partial \lambda}\, d\mu\, d\nu$

The flux through the pair of surfaces at right angles to ds_λ is therefore

$$\frac{\partial}{\partial \lambda}\left(\frac{h_1}{h_2 h_3}\frac{\partial \varphi}{\partial \lambda}\right) d\mu\, d\nu\, d\lambda \tag{2.9}$$

The volume of the space element is

$$d s_\lambda\, d s_\mu\, d s_\nu = \frac{d\lambda\, d\mu\, d\nu}{h_1 h_2 h_3} \qquad \text{[see (2.5)]}$$

Hence, with similar expressions for the other pairs, the equation of continuity becomes:

$$\nabla^2\varphi = h_1 h_2 h_3 \left[\frac{\partial}{\partial \lambda}\left(\frac{h_1}{h_2 h_3}\frac{\partial \varphi}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{h_2}{h_1 h_3}\frac{\partial \varphi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_3}{h_1 h_2}\frac{\partial \varphi}{\partial \nu}\right)\right] \tag{2.10}$$

Inserting values (2.6) for h_1, h_2, and h_3 and putting:

$$\left.\begin{aligned} L &= \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \\ M &= \sqrt{(a^2+\mu)(b^2+\mu)(c^2+\mu)} \\ N &= \sqrt{(a^2+\nu)(b^2+\nu)(c^2+\nu)} \end{aligned}\right| \tag{2.11}$$

We obtain

$$\left.\begin{aligned} \nabla^2\varphi = -\frac{4}{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)} \Big[(\mu-\nu)\, L \frac{\partial}{\partial \lambda} L \frac{\partial \varphi}{\partial \lambda} + \\ + (\nu-\lambda)\, M \frac{\partial}{\partial \mu} M \frac{\partial \varphi}{\partial \mu} + (\lambda-\mu)\, N \frac{\partial}{\partial \nu} N \frac{\partial \varphi}{\partial \nu}\Big] \end{aligned}\right| \tag{2.12}$$

3. Solution for the Motion Parallel to a Principal Axis. The special solutions found for the ellipsoid of revolution suggest trying to find a solution for the motion of an ellipsoid in a perfect fluid parallel to the x axis having the form $\quad \varphi = x\,\chi \quad$ **(3.1)**
where χ is a function of λ only. A comparison of such solution with the boundary condition will then show whether this will lead to a correct result.

We compute the divergence of (3.1)

$$\frac{\partial \varphi}{\partial x} = \chi + x \frac{\partial \chi}{\partial x}\,; \quad \frac{\partial^2 \varphi}{\partial x^2} = 2\frac{\partial \chi}{\partial x} + x \frac{\partial^2 \chi}{\partial x^2}$$

$$\nabla^2 \varphi = 2\frac{\partial \chi}{\partial x} + x \nabla^2 \chi \tag{3.2}$$

Now $\quad \dfrac{2}{x}\dfrac{\partial \chi}{\partial x} = \dfrac{2}{x}\dfrac{\partial \chi}{\partial \lambda}\dfrac{\partial \lambda}{\partial x}$ and by (2.7)

$$= \frac{2}{x} h_1^2 \frac{\partial \chi}{\partial \lambda}\frac{\partial x}{\partial \lambda} \quad \text{and by (2.4)}$$

$$= \frac{1}{a_2 + \lambda} h_1^2 \frac{\partial \chi}{\partial \lambda} \quad \text{and finally by (2.6)}$$

$$\frac{2}{x}\frac{\partial \chi}{\partial \lambda} = \frac{4\,(b^2+\lambda)\,(c^2+\lambda)}{(\lambda-\mu)\,(\lambda-\nu)}\frac{\partial \chi}{\partial \lambda}$$

Then by (2.12), applying this form to χ as a function of λ only,

$$\nabla^2 \chi = \frac{-4L}{(\nu-\lambda)\,(\lambda-\mu)}\frac{\partial}{\partial \lambda}\left(L \frac{\partial \chi}{\partial \lambda}\right)$$

Then (3.2) becomes, in elliptic coordinates,

$$L \frac{\partial}{\partial \lambda}\left(L \frac{\partial \chi}{\partial \lambda}\right) = -(b^2+\lambda)\,(c^2+\lambda)\frac{\partial \chi}{\partial \lambda} \tag{3.3}$$

This is satisfied by
$$\chi = A \int_\lambda^\infty \frac{d\lambda}{L\,(a^2+\lambda)} \tag{3.4}$$

giving
$$\varphi = A\,x \int_\lambda^\infty \frac{d\lambda}{L\,(a^2+\lambda)} \tag{3.5}$$

and
$$\frac{\partial \varphi}{\partial \lambda} = A \frac{\partial x}{\partial \lambda}\int_\lambda^\infty \frac{\partial \lambda}{L\,(a^2+\lambda)} - \frac{A\,x}{L\,(a^2+\lambda)} \tag{3.6}$$

Hence $\dfrac{\partial \varphi}{\partial \lambda}$ for $\lambda = 0$ becomes

$$\left(\frac{\partial \varphi}{\partial \lambda}\right)_{\lambda=0} = A\,x\left[\frac{1}{2a^2}\int_0^\infty \frac{d\lambda}{L\,(a^2+\lambda)} - \frac{1}{a^3 b c}\right] \tag{3.7}$$

and
$$\frac{\partial \varphi}{\partial x} = A \int_0^\infty \frac{d\lambda}{L\,(a^2+\lambda)} + \frac{A\,x \dfrac{\partial \lambda}{\partial x}}{L\,(a^2+\lambda)}$$

hence for $x = 0$
$$\frac{\partial \varphi}{\partial x} = A \int_0^\infty \frac{d\lambda}{L\,(a^2+\lambda)} \tag{3.8}$$

We introduce for abbreviation

$$\left.\begin{aligned} \alpha &= a\,b\,c \int_{\lambda}^{\infty} \frac{d\lambda}{L\,(a^2+\lambda)} \\ \beta &= a\,b\,c \int_{\lambda}^{\infty} \frac{d\lambda}{L\,(b^2+\lambda)} \\ \gamma &= a\,b\,c \int_{\lambda}^{\infty} \frac{d\lambda}{L\,(c^2+\lambda)} \end{aligned}\right| \tag{3.9}$$

We proceed now to establish the boundary condition in order to examine whether (3.4) is the correct solution, and if so, to determine the value of the constant multiplier A.

The solid ellipsoid in question may be $\lambda = 0$. Its equation is accordingly

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The motion may be parallel to the x axis, with the velocity U. The surface condition is then $\dfrac{\partial \varphi}{\partial \lambda} = -U \dfrac{\partial x}{\partial \lambda}$

which by (2.4) for $\lambda = 0$ becomes

$$\frac{\partial \varphi}{\partial \lambda} = -\frac{U\,x}{2\,a^2} \tag{3.10}$$

Combining (3.7) and (3.10) we obtain the boundary condition

$$\frac{A}{a^2}\left[\frac{1}{2}\,\frac{\alpha_0}{a\,b\,c} - \frac{1}{a\,b\,c}\right] = -\frac{U}{2\,a^2}$$

giving

$$A = \frac{a\,b\,c}{2-\alpha_0}\,U \tag{3.11}$$

where

$$\alpha_0 = a\,b\,c \int_{0}^{\infty} \frac{d\lambda}{L\,(a^2+\lambda)} \tag{3.12}$$

4. Discussion of the Solution. The velocity at points on the surface of the ellipsoid is again equal to the component of the maximum velocity. Hence the arguments given in VII 6 hold for any ellipsoid.

The apparent mass is computed from the kinetic energy. The latter is computed as in VII 7 from the ratio of the maximum velocity to the velocity of motion. The maximum velocity is given by (3.8). Hence the additional apparent mass results

$$\frac{\alpha_0}{2-\alpha_0}\,\varrho \cdot \mathrm{Vol.} \tag{4.1}$$

and the coefficient of apparent mass

$$k = \frac{\alpha_0}{2-\alpha_0} \tag{4.2}$$

The most general translational motion of the ellipsoid can be treated by the combination of the solutions of this chapter obtained by

exchanging the axes and a, b, and c, correspondingly. The flows superpose, and the resulting maximum velocity is in general no longer parallel to the motion, but remains constant at all intersection points of the ellipsoid with a plane through its center, that is, at all points of a "greatest ellipse" of the ellipsoid. At all other points of the surface the velocity is the component of this maximum velocity.

Except when moving parallel to a principal axis, there occurs a resulting couple of the fluid forces. When moving parallel to a principal axis, the motion is unstable except when moving parallel to the smallest axis.

5. Evaluation of the Constants. The values of the integrals (3.9) cannot be found by elementary functions, but constitute a class known as elliptical integrals. These can at best be reduced to certain "elliptical normal integrals" which are to be found tabulated in the literature of the subject. This reduction can be performed directly, but more conveniently by the use of the so-called elliptic functions.

A few remarks on elliptic functions will enable the reader to follow the general method. Elliptic functions are generalizations of the ordinary sine and cosine. They differ from them in that there is a triplet of three related functions rather than a pair of two. Besides the argument, u, of which they are the function, they have a parameter, called the modulus and denoted by k, which is implied the same for different elliptic functions occurring in the same equation. If this modulus k becomes zero, the two primary elliptic functions degenerate into the ordinary sine and cosine, and the third becomes constant and equal to one.

The three elliptic functions are:

sn u, called *sinus amplitude*, *cn u*, called *cosinus amplitude* and *dn u*, called *delta amplitude*. They are connected with each other by the equations

$$\left.\begin{aligned} sn^2 u + cn^2 u &= 1 \\ dn^2 u - k^2 cn^2 u &= 1 - k^2 \\ dn^2 u + k^2 sn^2 u &= 1 \end{aligned}\right\} \qquad (5.1)$$

It is easily seen that by putting $k = 0$ we obtain the corresponding equations for sine and cosine. They possess simple addition theorems $sn\,(u_1 + u_2)$ etc. which will not be needed. We shall use differentiation theorems, derived from the addition theorems, which are

$$\left.\begin{aligned} \frac{d\,sn\,u}{du} &= cn\,u\,dn\,u \\ \frac{d\,cn\,u}{du} &= -\,sn\,u\,dn\,u \\ \frac{d\,dn\,u}{du} &= -\,k^2\,sn\,u\,cn\,u \end{aligned}\right\} \qquad (5.2)$$

The elliptic functions were developed later than the tables of normal integrals used in connection with them, and for this reason the tables fail to correspond directly to these functions. Two tables are needed,

with two entrances, *viz*, for the variable u and the parameter or modulus k. The value, given in the first table commonly used is not $sn\, u$ as a function of u for values of the parameter, k, but u is given as $sn^{-1}(x)$ for values of the mod. k. That is, we may write

$$x = sn\,(u)\ mod.\ k$$

or shorter

$$x = sn\,(u, k)$$

but the table contains

$$F \doteq u = F\,(x, k) = sn^{-1}\,(x)\ mod.\ k = sn^{-1}\,(x, k) \tag{5.3}$$

Furthermore, many tables, instead of giving x and k, give only k and instead of x, give an angle φ so that $sin\, \varphi = x$. Some tables substitute besides an angle α for k, in the same way, $k = sin\, \alpha$. This F or u is called the normal elliptic integral of first kind. It is defined by

$$u = F\,(k, x) = F\,(k, \varphi) = \int_0^x \frac{d\,x}{\sqrt{(1 - x^2)\,(1 - k^2\, x^2)}} = \int_0^\varphi \frac{d\,\varphi}{\sqrt{1 - k^2\, sin^2\, \varphi}} \tag{5.4}$$

If $x = 1$, that is $\varphi = \pi/2$, the integral is called complete and denoted by $\boldsymbol{F}$ or $\boldsymbol{K}$.

The normal elliptic integral of second kind is

$$u = E\,(k, x) = (E\, k, \varphi) = \int_0^x \frac{\sqrt{(1 - k^2\, x^2}\, d\,x}{\sqrt{1 - x^2}} = \int_0^\varphi \sqrt{1 - k^2\, sin^2 \varphi}\; d\,\varphi \tag{5.5}$$

This function shows the same variations in the use of the variables, and they are denoted by the same symbols. This is not an inverse elliptic function but is connected with them by means of the equation

$$E\,(k, \varphi) = F\,(k, \varphi) - k^2 \int_0^u sn^2\, u\, du \tag{5.6}$$

so that

$$F\,(k, \varphi) - E\,(k, \varphi) = k^2 \int_0^u sn^2\, u\, du \tag{5.7}$$

The complete integral, for $x = 1$, is denoted by $\boldsymbol{E}$.

$\boldsymbol{E}$ and $\boldsymbol{F}$, the complete integrals, and $E\,(k, \varphi)$ and $F\,(k, \varphi)$ can be looked up in the tables.

After these remarks on elliptic functions and elliptic integrals, we proceed to transform the integrals (3.9) into expressions containing normal integrals only.

We assume $a > b > c$ and choose for the mod. k

$$k = sin\, \alpha = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \tag{5.8}$$

As sinus amplitude we choose in the same way

$$x = sin\, \varphi = \sqrt{\frac{a^2 - c^2}{a^2 + \lambda}} \tag{5.9}$$

so that $sn\,(u, k)$ or shorter $sn\,(u)$ becomes

$$sn\,u = \sqrt{\frac{a^2 - c^2}{a^2 + \lambda}} \tag{5.10}$$

The values of $cn\,u$ and $dn\,u$ are then likewise algebraic functions of a, b, c, and λ, computed from (5.8) and (5.10) by means of (5.1).

As the first step we have now to substitute for the integrals (3.9) integrals containing u and its elliptic functions with respect to the mod. k as given by (5.8). As the second step, we have to substitute the normal integrals F and E for all other elliptic functions. The first substitution gives

$$\left.\begin{aligned} \alpha &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}} \int_0^u sn^2\,u\,du \\ \beta &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}} \int_0^u \frac{sn^2\,u}{dn^2\,u}\,du \\ \gamma &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}} \int_0^u \frac{sn^2\,u}{cn^2\,u}\,du \end{aligned}\right| \tag{5.11}$$

These expressions are obtained by writing as an intermediate step

$$\left.\begin{aligned} a^2 + \lambda &= \frac{a^2 - c^2}{sn^2\,u} \\ b^2 + \lambda &= (a^2 - c^2)\,\frac{dn^2\,u}{sn^2\,u} \\ c^2 + \lambda &= (a^2 - c^2)\,\frac{cn^2\,u}{sn^2\,u} \\ \frac{d\,\lambda}{L} &= -\frac{2\,d\,u}{\sqrt{a^2 - c^2}} \end{aligned}\right| \tag{5.12}$$

where as before L stands for $\sqrt{(a^2 + \lambda)\,(b^2 + \lambda)\,(c^2 + \lambda)}$.

These four equations are found by using (5.1), (5.8), and (5.10), which are sufficient for obtaining the first three of (5.12), and then (5.2) for the last, first expressing λ from (5.10) and then differentiating.

From (5.11) we obtain finally

$$\left.\begin{aligned} \alpha &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}}\,\frac{1}{k^2}\left[F\,(u) - E\,(u)\right] \\ \beta &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}}\,\frac{1}{k^2(1 - k^2)}\left[E\,(u) - (1 - k^2)\,F\,(u) - k^2\sqrt{\frac{(a^2 - c^2)\,(c^2 + \lambda)}{(a^2 + \lambda)\,(b^2 + \lambda)}}\right] \\ \gamma &= \frac{2\,a\,b\,c}{\sqrt{(a^2 - c^2)^3}}\,\frac{1}{1 - k^2}\left[\sqrt{\frac{(a^2 - c^2)\,(b^2 + \lambda)}{(a^2 + \lambda)\,(c^2 + \lambda)}} - E\,(u)\right] \end{aligned}\right\} \tag{5.13}$$

where the mod. k and x in E and F are given by (5.8) and (5.9)

These last equations (5.13) may be verified by differentiating them, using as intermediate steps

$$\sqrt{\frac{(a^2-c^2)(c^2+\lambda)}{(a^2+\lambda)(b^2+\lambda)}} = \frac{sn\,u\;cn\,u}{dn\,u}$$

$$\sqrt{\frac{(a^2-c^2)(b^2+\lambda)}{(a^2+\lambda)(c^2+\lambda)}} = \frac{sn\,u\;dn\,u}{cn\,u}$$

and further (5.1), (5.2), and (5.7).

It is possible to transform (3.9) directly into (5.13) by mere algebra, using (5.4) and (5.5), or at least to verify the transformation in that way. The introduction of elliptical functions first, followed by a return to integrals, may look like a detour. The direct way is, however, more elaborate than the method adopted. The values of α_0, β_0 and γ_0 may now be computed from (5.13) by putting $\lambda = 0$.

6. The Elliptic Disc. The apparent mass of the elliptic disc is of interest in the wing theory, because according to III (4.6) the unstable moment of an elliptic wing is proportional to an expression containing its apparent mass in two directions. If the aspect ratio is large, it is generally assumed that the moment corresponds to the two-dimensional flow, the apparent mass then being equal to the mass of an ellipsoid of revolution with the span as axis and the chords as diameter and of the same density as the fluid. This volume is $(4/3)\,\pi\,a\,b^2$.

It is of interest to determine the magnitude of the error committed in neglecting the effect of the aspect ratio. This error is given by the difference between 1 and the coefficient of the apparent mass relative to the above volume.

TABLE 2.
Inertia factors of elliptic discs. $K = k\,b\,a^2\,\pi/6$.

b/a	Aspect ratio	Inertia factor k	b/a	Aspect ratio	Inertia factor k
∞	∞	1			
14.30	18.20	0.991	6.00	7.64	0.964
12.75	16.20	0.987	5.02	6.40	0.952
10.43	13.30	0.985	4.00	5.09	0.933
9.57	12.20	0.983	3.00	3.82	0.900
8.19	10.40	0.978	2.00	2.55	0.826
7.00	8.91	0.972	1.00	1.27	0.637

The coefficient of apparent mass relative to the volume zero of the disc becomes infinite, and we have to transform the formula of the apparent mass to make it applicable to this special case. Starting from

$$\gamma = \frac{2\,a\,b\,c}{\sqrt{(a^2-c^2)^3}}\,\frac{(a^2-c^2)}{(b^2-c^2)}\left[\sqrt{\frac{(a^2-c^2)(b^2+\lambda)}{(a^2+\lambda)(c^2+\lambda)}} - E\,(u,k)\right] \tag{6.1}$$

we put $\lambda = 0$, $c = c_0$ where c_0 is infinitely small of first order. This gives

$$\gamma_0 = 2\left[1 - \frac{c_0}{b}\,E\right] \tag{6.2}$$

The volume of apparent mass of the ellipsoid for this case is

$$K_3 = \frac{4\pi}{3} \cdot a\, b\, c \cdot \frac{\gamma_0}{2-\gamma_0}$$

giving for the disc, $c = 0$.

$$K_3 = \frac{4\pi}{3} a\, b^2 \cdot \frac{1}{E}$$

Again $\varphi = \pi/2$, so that E is the complete integral and the modulus $k = \sqrt{1 - \frac{b^2}{u^2}}$ hence $\alpha = cos^{-1}\ (b/a)$. Hence the ratio of this volume to the volume of the ovary ellipsoid of revolution with the disc as meridian is $1/E$ where E is the complete elliptic integral with the modulus $k = sin\, \alpha$ and $cos\, \alpha = b/a$.

This factor of apparent mass has been computed and is given in Table 2 for several values of the ratio of the largest to the smallest axis, and the ratio of the area of the disc to the square of its largest diameter, the so-called aspect ratio, has likewise been inserted. Table 2.

7. Rotation of an Ellipsoid. The rotation of an ellipsoid with three different axes in a perfect fluid otherwise at rest about one of its principal axes can be treated in a similar manner as with the problem of VII 10. As there is no present prospect of applying this case to the solution of aeronautical problems, a short indication of the method will suffice.

We start with a tentative solution of the form

$$\varphi = y\, z\, \chi \tag{7.1}$$

where χ is again a function of λ only. Forming the divergence of (7.1) gives

$$\nabla^2 \chi + \frac{2}{y} \frac{\partial \chi}{\partial y} + \frac{2}{z} \frac{\partial \chi}{\partial z} \tag{7.2}$$

From (2.7) we then have

$$\frac{2}{y} \frac{\partial \chi}{\partial y} + \frac{2}{z} \frac{\partial \chi}{\partial z} = 2\, h_1^2 \left(\frac{1}{y} \frac{\partial y}{\partial \lambda} + \frac{1}{z} \frac{\partial z}{\partial \lambda} \right) \frac{\partial \chi}{\partial \lambda} \tag{7.3}$$

This with (2.4) and (2.6) gives the form:

$$\frac{2}{y} \frac{\partial \chi}{\partial y} + \frac{2}{z} \frac{\partial \chi}{\partial z} = 4 \frac{L^2}{(\lambda-\mu)(\lambda-\nu)} \left(\frac{1}{(b^2+\lambda)} + \frac{1}{(c^2+\lambda)} \right) \frac{\partial \chi}{\partial \lambda}$$

whence

$$\frac{\partial}{\partial \lambda} log \left[L \frac{\partial \chi}{\partial y} \right] = - \left(\frac{1}{(b^2+\lambda)} + \frac{1}{(c^2+\lambda)} \right) \tag{7.4}$$

A solution of this equation is

$$\chi = A \int_{\lambda}^{\infty} \frac{d\lambda}{L\,(b^2+\lambda)\,(c^2+\lambda)} \tag{7.5}$$

where L is the same as in (2.11). The surface condition is

$$\frac{\partial \varphi}{\partial \lambda} = \Omega_x \left(z \frac{\partial y}{\partial \lambda} - y \frac{\partial z}{\partial \lambda} \right)$$

This is satisfied by $$\varphi = A\, y\, z \int_{\lambda}^{\infty} \frac{d\,\lambda}{L\,(b^2+\lambda)\,(c^2+\lambda)} \tag{7.6}$$

if the constant multiplier A is taken as

$$A = \frac{(b^2-c^2)^2\, a\, b\, c\, \Omega_x}{2\,(b^2-c^2)+(b^2+c^2)\,(\beta_0-\gamma_0)}$$

where β_0 and γ_0 come from (3.9) by putting $\lambda = 0$.

The kinetic energy is computed from III (2.1) giving

$$2\,T = \varrho\, A\, \Omega_x \int_0^{\infty} \frac{d\,\lambda}{L\,(b^2+\lambda)\,(c^2+\lambda)} \iint (n\,y - m\,z)\, y\, z\, d\,S$$

where m and n are the y and z direction cosines of the normals to the ellipsoid. This gives

$$2\,T = \frac{\Omega^2}{5}\,\frac{4}{3}\,\pi\,\varrho\, a\, b\, c\, \frac{(b^2-c^2)^2\,(\gamma_0-\beta_0)}{2\,(b^2-c^2)+(b^2+c^2)\,(\beta_0-\gamma_0)}$$

8. Concluding Remarks. The potentials (3.5) and (7.6) can be derived from a potential φ', thus

$$\varphi_1 = \frac{\partial\,\varphi'}{\partial\,\chi} \qquad \varphi_2 = y\,\frac{\partial\,\varphi'}{\partial\,z} - z\,\frac{\partial\,\varphi'}{\partial\,y}$$

where φ_1 is the potential of (3.5) and φ_2, that of (7.6). The potential φ' occurs in gravitational theory and represents the gravity potential of the solid ellipsoid. It is

$$\varphi' = a\, b\, c\, \pi \int_{\lambda}^{\infty} \left[\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda}\right] \frac{d\,\lambda}{L}$$

The three potentials of the translational motion parallel to the principal axes a, b, and c are obtained by differentiating the same function, φ', but multiplied by different constant multipliers.

This relation permits of certain conclusions regarding simple distributions of doublets giving rise to the flow produced by the straight motion of the ellipsoid. The same flow serves for all "confocal" ellipsoids, including the elliptic disc in the $a\,b$ plane (the two larger principal axes) with semi-axes $\sqrt{a^2-c^2}$ and $\sqrt{b^2-c^2}$. The doublets are arranged over this disc with a distribution of intensity the same as that of the mass density of an infinitely thin ellipsoid over the disc.

Bibliography.

For an extended bibliography on the general subject of Fluid Mechanics, the reader is referred to

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DIVISION D

HISTORICAL SKETCH

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PREFACE

In the development of aerodynamics three main periods can be distinguished: a first period, from antiquity to the publication of Newton's Principia (1687), a second period, from that epoch to the paper read by Lanchester before the Birmingham Natural History and Philosophical Society (1894), and a third period, from that epoch to the present time.

In the first period, philosophic thought on such matters was chiefly concerned with the mutual actions between a fluid medium and bodies which move through it. In the second period, aside from the development of experimental work, ever more accurate and more highly organized, a remarkable mathematical theory, based on an ideal fluid was created. This theory, however, lacked in direct applicability to the practical problems of actual fluids, such as air and water. In the third period, under the stimulus of the practical realization of flight, this theory, supplemented in some respects, seems to be well on the way toward furnishing an explanation of the outstanding experimental phenomena, and if not yet fully, it does, nevertheless, furnish a useful effective basis for prediction and control over much of the field of practical aeronautics.

In these three periods we meet with many prominent figures: in the first, Aristotle, Leonardo da Vinci, Galileo Galilei; in the second, Newton, Daniel Bernoulli, Euler, Lagrange, d'Alembert, Rankine, Stokes, Helmholtz, Kirchhoff, Rayleigh, W. Thompson (Lord Kelvin); in the third, Lanchester, Joukowski, Kutta, Prandtl, Kármán and others.

Around these principal names the present historical sketch will be developed, attempting to put in evidence the part which they and their disciples have taken in the development of this branch of science.

CHAPTER I

PERIOD OF EARLY THOUGHT: FROM ANTIQUITY TO THE END OF THE XVII CENTURY

The opinions of the ancients in this, as in other fields of science, are found summed up and set forth in a definitive manner by Aristotle (384—322 B. C.), whose conclusions were officially accepted up to the times of Galileo and Bacon: the two great founders of the experimental method.

Aristotle, however, dealt with the problem of the mutual actions between a body and a fluid through which it moves, only in an indirect way, dedicating to the subject only a few brief statements; but these few statements supplied matter of comment and argument for twenty centuries.

The passage in question is found in Section eight of the Fourth Book of his "Physica", which section was intended to demonstrate that a vacuum cannot exist. Now an argument to demonstrate this impossibility was found by Aristotle in a certain propulsive action which he assumed in the air in order to explain the motion of projectiles.

Here for the moment we must abandon our modern mode of thought, subsequent to the formulation of the law of inertia and our knowledge of the effects of friction, and remember that the ancients, in order to conceive of a body in motion, found it necessary to imagine a continous application of force to such body, and in default of which the body would immediately come to rest.

As regards the manner of application of this propulsive force to the body, it was conceived of as acting only through direct contact. That is, a body to be able to move had to be in contact with another body in motion, which in turn had to be in contact with another body in motion, and so on. This was a general law which even the celestial spheres obeyed, each of which was supposed to be maintained in motion by another sphere in immediate contact with it, until the last one (that of the fixed stars) should receive directly its motion from the Prime Mover, from whom all motions in the world were thought to be derived through a very long but finite series of intermediate transmissions.

From this law, of course, projectiles could not escape, notwithstanding that here there was no visible vehicle carrying them from the moment when they had left the body with which they were before in contact, until they reached their target.

But notwithstanding that the vehicle was not visible, it existed nevertheless, and was the air. From this Aristotle deduced that, without air, motion of a projectile would not be possible and, as a final step, a vacuum cannot therefore exist.

As regards the special manner in which air could exert its action on a projectile, Aristotle did not linger over this question. It was enough for him to have established the existence of this action; but as to its manner of working, he left the investigation of this matter to his successors, limiting himself only to say that two hypotheses were possible. One of these claimed the action of the air in closing violently to fill the vacuum, as furnishing a push on the projectile from the rear; the other assumed that air, by reason of its special fluidity, was able, if once put in motion by the body with which the projectile was previously in contact, to continue its action on the latter.

This doctrine of projectile motion founded on the action of the air, and generally of a *fluid medium* was in the VI century A. D. opposed, in the name of the evidence of the senses, by another doctrine proposed by the Greek grammarian Philoponus, who assumed that when casting a projectile, a certain energy or *impetus* was transfused into it by the caster, this energy being able to maintain the projectile in motion for more or less time.

These two doctrines, known respectively as the *medium* theory and the *impetus* theory did battle with each other for ten centuries, up to the discovery of the law of inertia. Following this, all controversy on this point ceased, and the air was conceived of as a factor of resistance only.

But before reaching this period, something should be said regarding Leonardo da Vinci (1452—1519), and his theories of flight—not because he contributed in any direct way to the development of the science, because his thoughts have remained unknown up to recent times; but in order to show the manner in which this man attacked and solved problems which have been attacked and solved otherwise only after four centuries.

Leonardo, as is well known, did not publish any book on his manifold researches, nor did he leave ordered and complete manuscripts, but only notes without order in booklets or on scattered sheets of paper, which in successive epochs have been collected, often in the most disordered manner in the form of various codices known under various names.

However his notes on aerodynamics have been recently collected and edited with comment and explanation[1], so that today it is possible to form a clear and complete idea of the thought of Leonardo on this matter.

Neglecting here his earliest researches in which, being still under the influence of the Aristotelian physics, he assumed an action of the air assisting motion, it is a matter of fact that at a certain epoch (1506) Leonardo abandoned completely these older ideas, recognizing in the air only a resisting action, the cause of which he ascribed essentially

[1] Giacomelli, R., "The aerodynamics of Leonardo da Vinci". *The Journal of the Royal Aeronautical Society*. vol. XXXIV, n. 240, London, December, 1930.

to its condensability (compressibility). He knew indeed that the operation of dividing the air and of putting it in movement constituted a part of the total resistance met by bodies in motion through the air, but he thought that this part was of little consequence in comparison with the frontal resistance due to the condensation of air under the pressure of the moving body.

Besides this resistance, Leonardo attributed to the condensation of the air also the cause of the lift of birds, deducing therefrom the possibility of human flight.

Air under the stroke of the wing condenses, Leonardo said, thus acquiring the properties of a solid body on which the bird is able to sustain itself; but to secure this condensation, a sufficient rapidity of wing stroke in necessary, and this rapidity must be such as to surpass that with which the layer of air stricken by the wing can transmit the stroke to the successive underlying layers of air. Under such conditions, air condenses locally at the instant, and becomes able to support the bird, which thus glides on the air as on an inclined plane.

It can be seen from this that Leonardo applied to the usual velocities of flight results by way of air condensation which modern aerodynamics applies only to very high velocities.

But Leonardo continued his reasoning, saying that it is not at all necessary that the wing should strike "motionless" air in order to obtain lift: the effect is the same if air in motion strikes the motionless wing. That which matters indeed is only a relative velocity of sufficient intensity between the two: air and wing. When this condition is accomplished, flight is possible. Flight is thus not absolutely dependent on the working of a wing; the effect of the latter can be produced by the movement of the air. Flapping and soaring flight were thus reduced by Leonardo to the same principle and the possibility for man of solving the problem of mechanical flight, under the form of soaring (the only form possible with his limited muscular energy) was definitely stated by Leonardo in 1505.

We now pass to Galileo (1564—1642) who dedicated the whole second day of his "Dialogues on the Principal Systems" (1632) to combat the Aristotelian theory of the *medium*, and to demonstrate the resisting action of air upon bodies moving through it. The capital merit of Galileo, as is known, was that of having succeeded in recognizing the positive and essential elements of the phenomena presented by a body in motion and of separating out the negative influence of friction and resistance, thus reaching the formulation of the persistence of motion: a fundamental law, which marked the beginning of modern mechanics.

But Galileo, though having demonstrated the resisting action of the air in the phenomena of motion, entered into a quantitative estimate as to this resistance only in a short passage of the fourth day of his

dialogues, entitled "Discourses and Mathematical Demonstrations Regarding two New Sciences, Belonging to Mechanics and Local Movements" (1638). The question which he asks is as to how the resistance of the air varies with the velocity.

To solve this problem Galileo made the following experiment: Having suspended two equal lead spheres with two cords of equal length, he caused them to oscillate with an amplitude respectively of 10 deg. and 160 deg., so that their velocities (as he believed) were in the ratio of 1:16. By measuring the oscillation numbers at the end of a certain time, "we see", said Galileo, "that the two numbers are equal, which is the proof that the two pendulums have been resisted by air proportionally to their velocities".

A very important consequence of this result was, according to Galileo, that the path of a projectile, calculated for the vacuum condition, held good also in air, its form not being altered by air resistance.

We are now in a position to know that the experimental device of Galileo was not appropriate for this purpose, and that in any event his extension to projectile velocities, of results obtained while experimenting at very low velocities, was not justifiable.

It should be noted, however, that Galileo himself doubted whether beyond a certain limit, the law of proportionality between resistance and velocity would hold good. In fact some lines further on he said that he was not aware whether, for very high velocities, such as those of projectiles from firearms, such a law could be considered of any value, thinking rather that in such cases the path, calculated for a vacuum, would be somewhat different from that in air, so that the beginning of the parabola would be less inclined and curved than the end.

The same law was assumed, without any justification, by Mariotte in his "Traité de la percussion et le choc de corps" (1679) while Descartes in his "Principles of Philosophy" (1644) had stated an increase in the resistance of air with velocity, but in quite general terms, and without entering into any quantitative determination. In any event, the hypothesis of a simple proportionality between resistance and velocity seemed the most natural and was shared by all the earliest investigators.

Huyghens also began with this hypothesis in his studies on falling bodies, as he himself said at the end of his "Discours sur la cause de la pesanteur" (1690); but, afterwards he abandoned this hypothesis, substituting for it another which he found to be more in accord with experience; that is, the hypothesis of the proportionality of the resistance to the square of the velocity.

Now this law which Huyghens found experimentally, was derived by Newton, by deduction, for certain special fluid conditions, as will be seen in the following pages.

CHAPTER II

PERIOD OF CLASSIC HYDRODYNAMICS: FROM THE END OF THE XVII CENTURY TO THE END OF THE XIX CENTURY

Before entering on Newton's researches on fluid resistance note should be made regarding his deeper purpose in carrying on these studies.

This was in fact, to draw from them the proof that cosmic space was not occupied by a material continuous fluid, according to the Cartesian and Aristotelian philosophies, but that instead it was quite empty of matter. In fact if there were any fluid this must resist celestial bodies in their course, whereas, astronomical observations reveal no trace of any such resistance. Led by this philosophical need, Newton examined the nature of fluid resistance, demonstrating that there was a part of this resistance which could in no case disappear.

The entire second book of his work "Mathematical Principles of Natural Philosophy" (1687 first edition, 1713 second, and 1726 third edition) is dedicated to this study in which actual fluids as water, air, oil, and mercury are considered together with fluids quite hypothetical, although ,well defined by special properties.

Newton (1642—1727) began his study by stating that resistance depends on three factors, that is, on the density of the fluid, on the velocity, and on the shape of the body in motion. He observed also that in addition to the resistance dependent on the fluid density (that is, on inertia) there were two other forms of resistance: one dependent on the tenacity of the parts of the fluid, and the other dependent on the attrition between the body and the fluid. Finally he observed also that a small portion of the total ıesistance depended as well on the elasticity of the fluid: Newton's concept of elasticity, however, being somewhat complicated and different from that of the present day.

At any rate in Newton's concepts of tenacity and attrition, as well as in those of his immediate successors, we cannot expect to find the exact content of our modern concepts of viscosity and friction.

The resistance produced by tenacity and elasticity both of which Newton assumed constant, could only be very small, especially at high velocities, so that they could be, according to Newton, pratically neglected.

The resistance coming from attrition, though proportional to the velocity, might be, in particular circumstances (that is, for special fluids) neglected as well. The resistance coming from the inertia of matter, on the contrary, due to the fact that inertia constitutes the essential mechanical property of matter, could never be lacking or disappear. This resistance was found by Newton to be proportional to the square of the velocity, by the following argument (scholium at the end of Section I):

"In mediums void of all tenacity, the resistances made to bodies are in the duplicate ratio of the velocities. For by the action of a swifter body, a greater motion, in proportion to a greater velocity, is communicated to the same quantity of the medium, in a less time; and in equal time, by reason of a greater quantity of the disturbed medium, a motion is communicated in the duplicate ratio greater; and the resistance (by Law 2 and 3) is as the motion communicated." (Motte's translation.)

In conclusion, in the more general case, the resistance of a body moving through a fluid consisted—as stated in the scholium at the end of Section III—of three parts: a first part uniform, a second proportional to the velocity, and a third part proportional to the square of the same, the last being the most important.

After having considered the first two factors of resistance, that is, density (in addition to which also tenacity, attrition and elasticity), and velocity, the third factor to be examined is the shape of the body in motion. In this connection it is interesting to quote the words of Newton's disciple and editor of the third edition of the "Principles", Henry Pemberton, who in his book "A view of Sir Isaac Newton's Philosophy" (London 1728), on page 155, writes:

"In the next place our author is particular in determining the degree of resistance accompanying bodies of different figure; which is the last of the three heads, we divided the whole discourse of resistance into. And in this disquisition he finds a very surprizing and unthought of difference between free and compressed fluids. He proves, that in the former kind, a globe suffers but half the resistance which the cylinder, that circumscribes the globe, will do if it move in the direction of its axis. But in the latter he proves, that the globe and cylinder are resisted alike. And in general, that let the shape of bodies be ever so different, yet if the greatest section of the bodies perpendicular to the axis of their motion be equal, the bodies will be resisted equally."

The demonstration for *compressed* or continuous fluids was given by Newton in lemma 7 and its scholium (Section VII), while that for *free* or discontinuous fluids was given in the thirty-fourth proposition (Section VII), in which the famous sine square law is found, which served, at the beginning of the XIX century, to demonstrate mathematically the impossibility of flying and by reason of which Newton has been blamed for having delayed aviation at least for half a century.

Now the demonstration by the sine square law of the impossibility of flying and the blame thrown upon Newton on this account, first noted by Villamil[1] were but the effects of a misunderstanding which consisted in applying to the air deductions which were not valid for it, as Newton in this proposition did not consider air or any other actual fluid, but rather a rarefied and frictionless hypothetical medium.

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[1] Villamil, Col. R. de., "The Sine Square Law", Aeronautics, pp. 55, 56, London, February, 1913.

In the title of the proposition it is said that it is, "a rare medium consisting of equal particles, freely disposed at equal distances from each other". (Motte's translation.)

In this hypothetical medium, Newton supposed a sphere, and the cylinder circumscribed about it, to move with an equal velocity in the direction of the axis of the cylinder. The question proposed was then to determine the ratio of the resistance of the sphere to that of the cylinder.

As the result of his research, Newton found, in fact, that the resistance of the sphere, the surface of which (except at one point) meets the fluid obliquely is exactly one half that of the cylinder, the base of which meets the fluid everywhere perpendicularly.

But it is neither this result, nor the paradoxical one found by Newton for continuous fluids, which is important for us, but the fact put in evidence during the demonstration of the thirty-fourth proposition, that the action of the fluid, on a plane inclined to the relative motion is equal to that exerted on a plane orthogonal to that direction, multiplied by the square of the sine of the angle of incidence. It should be especially noted that this demonstration is made by neglecting friction completely.

This is the famous proposition which has been made responsible for so much harm, which might have been avoided by a more careful reading of Newton's original text. Let us for instance read the scholium commenting on the following thirty-fifth proposition, in which the value of the resistance of the sphere is determined in case of the same hypothetical discontinuous fluid, whether elastic or inelastic. In fact in this scholium Newton states explicitly that his reasoning relates only to a fluid consisting "of very small quiescent particles of equal magnitude and equally disposed at equal distances from one another", while in the case of continuous fluids, resistance is less; and precisely, the resistance passing from discontinuous elastic fluids to inelastic ones diminishes by one half, and passing finally to continuous fluids the same resistance diminishes by one half again.

"I have exhibited in this proposition the resistance and retardation of spherical projectiles in mediums that are not continued .... if so be the globe and the particles of the medium be perfectly elastic, and are endued with the utmost force of reflexion; and that this force where the globe and particles of the medium are infinitely hard and void of any reflecting force, is diminished one half. But in continued medium, as water, hot oil, and quicksilver, the globe as it passes through them does not immediately strike against all the particles of the fluid that generated the resistance made to it, but presses only the particles that lie next to it, which press the particles beyond, which press other particles, and so on; and in these mediums the resistance is diminished one other half. A globe in these extremely fluid mediums meets with a resistance that is .... " (Motte's translation.)

The difference between discontinuous and continuous, or" compressed", fluids, and the reason why resistance diminishes in the latter

fluids, and why no connection exists in them between the shape of a body in motion and its resistance, is explained by Pemberton as follows:

„in the case of discontinued fluids the body by pressing against their particles, drives them before itself, while the space behind the body is left empty. But in fluids which are compressed, so that the parts of them removed out of place by the body resisted immediately retire behind the body, and fill that space, which in the other case is left vacant, the resistance is still less".

It is important to note that in the demonstration of the thirty-fourth proposition, Newton, almost incidentally, asserts in the clearest way the reciprocity principle (which had however been recognized already by Leonardo da Vinci) deducing it from a corollary of his Laws of Motion:

"For since the action of the medium upon the body is the same (by Cor. 5 of the Laws) whether the body moves in a quiescent medium, or whether the particles of the medium impinge with the same velocity upon the quiescent body; let us consider the body as if it were quiescent, and see with what force it would be impelled by the moving medium." (Motte's translation.)

Another very important and well known statement of Newton's, is found at the beginning of Section IX in the following form:

"Hypothesis

The resistance arising from want of lubricity in the parts of a fluid is *caeteris paribus*, proportional to the velocity with which the parts of the fluid are separated from each other." (Motte's translation.)

We have here, in this hypothesis, the first enunciation of the friction law for laminar flow stating that the frictional force depends on the relative velocity of adjacent layers.

After this rapid survey of the theory, mention must be made of the experiments carried out by Newton, both with pendulums and with bodies falling perpendicularly through water and air, to verify the same theory. His experiments with pendulums are described in the general scholium at the end of Section VI, and these with falling bodies in the scholium at the end of Section VII[1].

Pemberton dedicates only a few words in his book to these experiments, stating however that "all agree with the theory"; and concludes his account of both theory and experiments by Newton on fluid resistance, with the following words:

[1] A detailed account of these experiments can be found in a recent German publication "Handbuch der Experimental-Physik, Hydro- und Aero-Dynamik. 2. Teil: Widerstand und Auftrieb", herausgegeben von Ludwig Schiller (Akademische Verlagsgesellschaft, Leipzig 1932), containing a valuable monograph by O. Flachsbart, entitled "Geschichte der experimentellen Hydro- und Aeromechanik, insbesondere der Widerstandsforschung".

The manuscript of the present monograph had been completed before the appearance of this work, but the exigencies of publication have not permitted such references to it in the body of the present work as might otherwise have been made. While the major fields of the two monographs are different, there is an overlapping territory, and the interested reader will find in the above noted work much of interest and value supplementary to the present monograph.

"By this theory of the resistance of fluids, and these experiments our author decides the question so long agitated among natural philosophers whether the space is absolutely full of matter. The Aristotelians and Cartesians both assert this plenitude; the Atomists have maintained the contrary. Our author has chose to determine this question by his theory of resistance, as shall be explained in the following chapter."

Now this explanation is that, even supposing space filled by a subtile fluid, consisting of very minute and smooth parts, thus removing all adhesion and friction between them, resistance can never disappear, because, the fluid always resists on account of the inertia (Pemberton, as his other contemporaries, translates this term by "inactivity") of its parts;

"unless we will suppose the matter of which this fluid is composed, not to be endued with the same degree of inactivity as other matter. But if you deprive any substance of the property so universally belonging to all other matter, without impropriety it can scarce be called by this name".

Unfortunately the question was not decided by the resistance theory of Newton as his opponents made the objection that the frontal resistance of the fluid medium, on account of its inertia, could be balanced by a push of the same fluid on the back, rushing in to fill the vacuum left by the body—a concept to which the old Aristotelian ideas must have given a strong support.

This objection, which anticipated by almost a century the Paradox of D'Alembert, was not, so far as we are aware, answered directly by Newton. It was, however, answered instead by Roger Cotes, certainly interpreting the thought of his Master. We read, in fact, in the preface to the second edition of the "Principles" written by Cotes and published in the year 1713, fourteen years before Newton's death, this important passage:

After having said that the resistance arising from *want of lubricity* is very small, except in particular fluids like oil and honey, while that arising from inertia cannot be diminished, he continues:

"Bodies in going on through a fluid communicate their motion (momentum) to the ambient fluid by little and little, and by that communication lose their own motion (momentum) and by losing it are retarded. Therefore the retardation is proportional to the motion (momentum) communicated; and the communicated motion (momentum) when the velocity of the moving body is given, is as the density of the fluid; and therefore the retardation or resistance will be as the same density of the fluid, nor can it be taken away, unless the fluid coming about to the hinder parts of the body restore the motion lost. Now this cannot be done unless the impression of the fluid on the hinder parts of the body be equal to the impression of the parts of the body on the fluid, that is unless the relative velocity, with which the fluid pushes the body behind is equal to the velocity with which the body pushes the fluid; that is, unless the absolute velocity of the recurring fluid be twice as great as the absolute velocity with which the fluid is driven forward by the body, which is impossible." (Motte's translation.)

Newton's researches on fluid resistance were continued among others by Johan Bernoulli (1667—1748) and by his son Daniel (1700—1783). The former in his work "Discours sur les lois de la communication du mouvement" (1727) dealt with the resistance on Newton's hypothesis of a discontinuous medium, arriving at a formula likewise inapplicable to reality; while in his "Hydraulics" (1742) he demonstrated that the motion of a fluid issuing from a cylindrical vessel, as it had been conceived by Newton in his research on resistance in a continuous medium, was not according to reality.

Much more important were the researches of the son, Daniel Bernoulli, to whom, among other things, is due the introduction of the name "Hydrodynamics" a term created by him to unite under a single name two sciences—hydrostatics and hydraulics—up to that time dealt with separately, but which he proposed to unite in his work, owing to the intimate connection which he recognized between the two subjects.

Regarding the problem of resistance, Daniel already in the year 1727 in the second volume of "Mémoires de Pétersbourg" proposed a formula for its measure and tried also by means of various hypotheses on the motion of the fluid particles, to establish a relation between the resistances of solids of various shapes, arriving at conclusions which afterwards he rejected in his "Hydrodynamics" (1738), his main work, and which was very highly esteemed by Lagrange. In this work all propositions are derived from one principle alone, as in the "Mécanique analytique" of Lagrange, with this difference, however, that in the latter the principle is that of virtual work and in the former of conservation of the *vires vivae*.

One of the most brilliant applications of this principle in Bernoulli's "Hydrodynamics" is found in the deduction of that famous theorem known under his name, which establishes a connection between pressure and velocity, that is between potential and kinetic energy, so that points of higher pressure are those of less velocity and points of maximum pressure have a velocity equal to zero.

This deduction was made by Bernoulli on the following hypothesis:

"After having supposed the fluid to be divided into layers perpendicular to the direction of motion, it must be admitted that the particles of any layer all move with the same velocity and in such a manner that everywhere the fluid velocity is inversely proportional to the section of the vase." And he adds: "This hypothesis is useable notwithstanding that it is understood however that the fluid along the walls of the vase moves more slowly and in the middle more rapidly, which is by reason of friction . . . . . but the error which can arise from these defects is very rarely noticeable."

The starting point of Bernoulli in his deduction was the case of water issuing through a small orifice in the bottom of a vessel filled to the height h above the level of the orifice. In such a case — to use modern terms—the kinetic energy of the water is equal to the work effected

by gravity along the height h, and the velocity is equal to that obtained by a body falling freely from that height. On the other hand the pressure with the water at rest, is equal to the height of a water column of the height h. The kinetic energy in question is the "*vis viva*" of Bernoulli and our principle of the conservation of energy is his principle of "conservation of *vires vivae*", which however, not to shock those who did not like the name of "*vis viva*", owing to the long discussions on that name, he preferred to replace with the expression: "aequalitas inter descensum actualem ascensumque potentialem", a proposition stating that the fluid is able to ascend to the height from which it had descended. The definitions of *ascensus potentialis* and *descensus actualis*, in Bernoulli's own words were as follows:

"The ascensus potentialis of a system, the single parts of which move with any velocity, means the vertical height to which the center of gravity of this system will reach, if all the single particles, having converted their motion upward are conceived to ascend with their own velocity as far as they can; the descensus actualis instead means the vertical height along which the center of gravity descends, when the single particles had reached a state of rest." ("Hydrodynamics" p. 30.)

The possibility of a fluid reaching the same height from which it had descended, had been established by Torricelli (1608—1647) in his book "Opera Geometrica" (1644), applying to fluids the axiom established by Galileo for bodies descending and ascending along inclined planes, and for pendulums.

Galileo's axiom—as Bernoulli himself said in the first section of his "Hydrodynamics"—had been put in a more general form by Huygens (1629—1695) in his researches on the oscillation center of a compound pendulum, stating that the center of gravity of a system of descending bodies rises to the same height from which it had descended.

"From this axiom" Bernoulli continued, "immediately follows the principle of conservation of *vires vivae*, which was also demonstrated by Huygens, and by which it is assumed that if any weights whatsoever begin to move on account of their gravity in whatever manner, the single velocities will be everywhere such that the sum of the products of their squares by their masses is proportional to the vertical height, along which descends the compounded center of gravity of the bodies, multiplied by the sum of the masses[1]."

"It is marvelous of what utility is this hypothesis in mechanical philosophy, which if anyone understood rightly, such a one was certainly my father, who here and there, but first of all in his 'Dissertatio Parisiis edita de legibus motuum in Tom. 2 Comm. Acad. Imp. Sc. Petrop', demonstrated it, and the same hypothesis has been used by me in my investigations of the laws of motion on account of gravity in fluids, for I assumed the velocities of the particles always to be such that each one, having moved upward to a state of rest, their common center of gravity ascends to its primitive height."

After that, Bernoulli observed, however, that in the actual case, owing to the presence of a certain subtile matter (ether) to which

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[1] The expression $(1/2)\, m V^2$ instead of $m V^2$ was adopted later.

a part of the *vires vivae* or of the *ascensus potentialis* is transmitted by bodies in motion, "a part of the ascensus potentialis is continually lost, which must be taken into account in making the calculations".

At any rate he assured us that this principle, when applied with care, put him in a position to discover many new theorems on the flow of water, which he does not know if through other methods could have been either demonstrated or found.

Besides the case of water issuing through a small orifice in the bottom of a vase, which as noted above, was his starting point, Bernoulli in the 12th section of his book named by him "Hydraulico-statica" ("where", he said "not as much pressure from velocity as velocity from pressure, if there is a small hole in the wall of the canal, can be deduced"), considered a more general case of his theorem.

In this case he took into consideration the work of pressures in a tube of various sections and therefore with varying stream velocity, and drew from it the consequence, experimentally verified, that in the narrower sections of the tube, underpressure in comparison with the surroundings can occur, so that water can be sucked through a tube opening into one of these sections.

Before closing this subject it must be noticed that the "Bernoulli theorem" was very little emphasized by its author in his exposition, so that the modern reader who expects to find it occupying a prominent place in Bernoulli's treatise, as in our modern ones, is disappointed.

In the year 1741 in vol. VIII of the "Mémoires de Pétersbourg" Bernoulli gave a method of determining the pressure exercised by a vein of fluid issuing from a vessel against a plane; but this method was criticised by D'Alembert (1717—1783), who, in the introduction to his 'Essai d'une nouvelle théorie de la résistance des fluides" (1752) says:

"However this may be, M. Daniel Bernoulli agrees that this theory of the pressure of a jet of fluid against a plane would be of no great utility for determining the pressure on a plane entirely immersed in the fluid, because the movement of the particles of the fluid is very different in the two cases. In effect, in the case where the jet strikes the plane the particles of the fluid on reaching the plane change direction in such manner that they move immediately parallel to the plane and slide along the plane following this latter direction. This could not occur when the plane is entirely immersed in a deep fluid, for, as soon as the particles of the fluid leave the forward surface of the plane on which they have glided, they find themselves pushed and brought back toward the after surface by the neighboring fluid in movement, to the right and to the left, so that their direction, parallel as it was to the plane, becomes perpendicular to it or at least makes with it a very great oblique angle as daily experience demonstrates. Now this reflux of the particles and the pressure which may result therefrom on the after surface must change the pressure experienced by the forward surface."

From this and other considerations on the work of his predecessors, D'Alembert deduced the conclusion that the theory of fluid resistance although dealt with by many great geometricians, was still very

imperfect in its essential elements, which reason had impelled him to deal again with such a matter and quite independent of his predecessors in such studies.

As the basis of his theory D'Alembert made only a single hypothesis on the nature of the fluid: an hypothesis—he said—which could be denied by no one, which is that a fluid consists of very little particles separate from each other and able to move freely. As regards the resistance which the body opposes to the fluid—he asserted—that this goes back to the general case of the resistance shown by whatever body when struck by another; that is, the resistance is equal to the lost momentum: and this could be determined by means of the equations of the motion already established for solid bodies.

Now in order to establish these equations in his "Traité de Dynamique" (1743) D'Alembert had dealt with all dynamical problems by means of a principle which he states as follows:

"As the motion of a body, when changed, may be considered as composed of the motion which it had in the first place and of a new motion which it has received, so likewise can the motion of the body in the first place be considered as composed of the new motion which it has taken and of another which it has lost."

Also the problem of the resistance of a body in a fluid was dealt with by D'Alembert by means of the same principle, saying that "it is also to the laws of the equilibrium between the fluid and the bodies that I am reducing the research of this resistance".

The first attempt to apply this principle to fluid motion was made by D'Alembert in the year 1744 in his "Traité de l'équilibre et des mouvements de fluides pour servir de suite au traité de dynamique", but the method was perfected by him in his "Essai sur la résistance des fluides" (1752), and further in his "Opuscules mathématiques" of the year 1768 (XXXIV Mémoire).

"But it must not be supposed that this research, although facilitated by this means, can be as simple as that of the communication of motion between two solid bodies. Let us suppose, in effect, that we had the advantage, of which we are deprived, of knowing the shape and mutual disposition of the particles which compose fluids: The laws of their resistance and of their action would, without doubt, become reduced to the known laws of motion; for the study of the motion communicated by one body to any number of particles which surround it, is only a problem of dynamics for the solution of which all the necessary mechanical principles are available. However, the greater the number of particles, the more difficult would it become to apply these principles in a convenient and simple manner. In consequence such a method would be of little use in the study of the resistance of fluids. But we are far indeed from having all the data necessary for the application of this method, not only are we ignorant of the form and arrangement of the particles of fluids, but we are ignorant also as to how these particles are pushed by the body and how they move among themselves. There is furthermore so great a difference between a fluid and a collection of solid particles that the laws of pressure and of equilibrium of fluids are very different from the laws of the pressure and equilibrium of solids."

Not being able then to apply to the investigation of fluid motion the properties of solids and having at hand regarding the same fluids only that little information furnished by "Hydrostatics", D'Alembert concluded:

"This ignorance, however, has not prevented the making of great progress in hydrostatics, for philosophers, being unable to deduce immediately and directly from the nature of fluids the laws of their equilibrium, they have reduced them at least to a single principle of experience, the equality of pressure in all directions, a principle which they have regarded (for lack of something better) as the fundamental property of fluids."

While developing his investigation on this principle, D'Alembert took into consideration two hypotheses: that a rectangular parallelepiped in a mass of fluid in equilibrium is in equilibrium also, and that a portion of fluid, while passing from one place to another, will maintain its volume unaltered when the fluid is incompressible, or will vary its volume according to a given law when the fluid is elastic. (Principle of Continuity.)

Deducing in such a way the resistance met by a body moving through a fluid, D'Alembert, already in his first work the "Traité de l'équilibre et du mouvement des fluides pour servir de suite au traité de dynamique" (1744) had met with that famous and unforeseen result of the resistance becoming equal to zero.

The same fact was established by him also in his "Essai d'une nouvelle théorie de la résistance des fluides" (1752) and finally with different demonstration but with the same results was given again in vol. V of his "Opuscules mathématiques" for the year 1768 [1]. In this last formulation, D'Alembert came to the following conclusion:

"I do not see then, I admit, how one can explain the resistance of fluids by the theory in a satisfactory manner. It seems to me on the contrary that this theory, dealt with and studied with profound attention gives, at least in most cases, resistance absolutely zero: a singular paradox which I leave to geometricians to explain."

D'Alembert carried out also extended experimental researches. A very numerous series of them, carried out in collaboration with l'Abbé Bossut and Condorcet on account of the French Government for the purpose of determining the resistance of ships in canals are found in a volume published in the year 1777 under the title "Nouvelles expériences sur la résistance des fluides", in the preface to which it is stated that the research of the impulse of a fluid against a plane in motion, or of the resistance which is experienced by a solid body dividing a fluid, is perhaps the most important problem of hydrodynamics, both on account of its difficulty and of its applications to naval architecture, to the construction of dykes, of hydraulic machines, etc.

[1] XXXIV Mém. "Paradoxe proposé aux géomètres sur la résistance des fluides", pp. 132—138.

As regards the conclusions of such researches they were as follows:

(1) the resistance of a fluid is sensibly proportional to the square of the velocity;

(2) the resistance is proportional to the surfaces for perpendicular planes;

(3) the rule that for oblique planes resistance varies with the sine square of the angle of incidence holds good only for angles between 50 and 90 deg. and must be abandoned for lesser angles;

(4) the influence of viscosity in water is extremely small, particularly if the velocity is somewhat high.

These results are confirmed in the "Traité théorique et expérimentale d'hydrodynamique" by the Abbé Bossut, one of the collaborators of D'Alembert in the aforesaid experiments: a treatise of which the first edition was published in the year 1771 and the third in 1792, and which was approved by Lagrange.

In this treatise it is interesting to notice how Bossut even in the first pages asserted that he was completely aware of the great defects of the usual theory, which in certain points and particularly in that of the sine square law was untenable indeed. But he added that he had been obliged to follow it for the following practical reasons.

(1) On account of its being very simple and very convenient for calculations, so that no other theory had been able to replace it when it is necessary to pass from theoretical considerations to practical applications.

(2) On account of its having served as the basis for several excellent works such as the "Scientia Navalis" by Euler, the "Traité du navire" by Douguer, the "Manoeuvre des Vaisseaux" by the same author, etc.

(3) On account of its being applicable without remarkable errors to the determination of the effect of hydraulic wheels driven by a water stream.

Also in an Italian treatise of wide circulation at that time "Elementi di Idraulica" (Fourth edition in the year 1826) the author, Venturoli, justifies himself for following the resistance theory based on the sine square law owing to its convenience in all practical cases of the calculation of the resistance of ships, of mill wheels, of dykes etc; cases in which the angle of incidence is not very far from 90 deg.

But Euler made use of the sine square law, not only in his Scientia Navalis of the year 1749, as noted by Bossut, but also in his successive researches, for example in that famous paper of the year 1763 presented to the Academy of Sciences of Pétersbourg, in which he entered into a discussion of the content of D'Alembert's paradox, so that this is known also under the name of Euler's paradox, although, as we shall see, he never accepted the possibility of a body moving in a fluid without meeting any resistance, but attributed this statement to a deficiency of the theory.

In this paper entitled "Delucidationes de Resistentia Fluidorum"[1] Euler (1707—1783) begins by observing that there were two methods for studying fluid resistance: the "usual method" according to Newton's theory, based on the hypothesis of resistance as the effect of the shock of the fluid, and a more scientific method, according to the hydrodynamic theory, based on the hypothesis of modern Geometricians of resistance as the effect of fluid pressure.

Euler continued saying that the shock hypothesis, leading to the sine square law, is far from the truth, as the fluid never strikes the body, but "before reaching the latter, bends its direction and its velocity so that when it reaches the body it flows past it along its surface, and exercises no other force on the body except the pressure corresponding to the single points of contact" which presure, according to the hydrodynamic theory, corresponds indeed to the diminution of *vis viva* of the fluid while passing from the undisturbed flow to that at the said points.

In connection with this statement it is interesting to note that Euler felt here the need of explaining the meaning of the statement regarding fluid pressure and fluid velocity (of the discoverer of which, Daniel Bernoulli, no mention is made by him) which is, that the more the fluid velocity at a point of a body is diminished, the more the fluid pressure at that point will be increased, a proposition which, if not rightly understood, can lead, according to Euler to a "great paradox" which consists in saying that "from a greater velocity a less resistance and from a less velocity a greater resistance will ensue". But the whole difficulty of understanding this proposition, as Euler said, disappears if we consider that here the comparison is not between the velocities of two different currents, for "it is quite sure that the more rapidly a fluid, or a body moves, the greater is the resistance," but rather the comparison is between the different velocities along a given current (vein of flow) which flows over the surface of a body[2].

Turning to the examination of the two methods for calculating fluid resistance Euler finds that notwithstanding its theoretical superiority the hydrodynamic method was far from being applicable in practice, so that one "cannot help using the first method whenever we need to calculate resistance, notwithstanding the fact that we are well aware of its insufficiency"; while on the other hand this first or usual method

[1] "Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae" vol. VIII, (anno 1763).

[2] The difficulty for beginners to understand Bernoulli's principle has been remarked also by other authors. For instance R. von Mises in his valuable elementary book "Fluglehre" (page 21 of the third edition, 1926) suggests to start always from the representation of a body at rest met by a current, as in the wind tunnel. It is then seen at once that "where the air particles are retarded and therefore have less velocity, there pressure increases".

certainly "deserves to be appreciated on account of its usefulness in calculations and still more on account of the fact that it has been found to be in practice not very far from the truth".

Having put the question in these terms Euler proposed to see if it was possible to improve the usual method, which we cannot help using, so as to make it more rapid and exact, which he thought to be possible by combining the two methods together.

To this end Euler wrote an equation in the left member of which there was the expression of the fluid resistance according to the hydrodynamic or pressure theory, and in the right there was the expression of the fluid resistance according to the usual or shock theory.

In fact, having indicated any point on the contour of a longitudinal section of the front half of a spheroid at rest in a current of water (or a spheroid moving through water at rest) by M, the angle of incidence of the tangent at M by φ and the velocity of the current at a great distance from the body (*in notabili a corpore distantia*) by K, Euler wrote the expression for the resistance at M according to the shock theory. Then, having indicated by V the velocity of the current at M, he wrote the expression of the resistance according to the pressure theory, and equated the two expressions:

$$K^2 \sin^2 \varphi = K^2 - V^2, \text{ whence } V = K \cos \varphi$$

The latter formula, according to Euler, made possible the determination of the *true velocity* V of the water at any point M of the front half of the body, having given the velocity K of the current at a great distance from the body.

But Euler observed that there are points on the surface of the body where the velocity of the current is equal to that at a great distance from it, and such points are those where the tangent becomes parallel to the direction of motion.

In fact if $\varphi = 0$, we have $V = K$.

Accordingly, the velocity of the current along the contour of the main section of a spheroid should equal the velocity of the current at a great distance from the body; a deduction indeed quite at variance with actual experience!

Euler then suggested a method for finding the velocity of a ship by measuring the water velocity at a point on the hull where the tangent is parallel to the direction of the motion: a method however, as he said, which is not perfectly exact, but approximate enough.

At any rate, as Euler concluded, the method could be applied only to the calculation of the fluid resistance at points on the front half of the body, that is, on the prow of a ship. This on account of the fact that the formula $V = K \cos \varphi$, giving the same velocity at corresponding points of the two halves of the body, the pressure resulted the same also, and "the stern of a ship would be propelled with the same force with

which the prow is repelled", a conclusion leading to the establishment of the principle of no resistance in perfect fluids, which at that time, according to the considerations of D'Alembert, was already admitted by some students, but which Euler always rejected.

In fact, Euler to avoid it, could find no better expedient than to state that his own formula $V = K \cos \varphi$ (derived from the Newtonian or usual method for calculating resistance) ceased to hold good from the main section of the spheroid backwards, owing to the fact that on the back of the spheroid, fluid resistance must certainly follow laws different from those for the front, so that he expressly warns the reader not to apply the usual method to the rear part of the bodies where, being no longer valid, it leads to erroneous conclusions. His final words in this connection are as follows:

"Therefore if some people attracted by the appearance of the said method, think that a body can be moved through a fluid meeting no resistance, as the force exercised on its front part by the fluid would be destroyed by the action of the same in the rear part (the which destruction they believe to be impeded in terrestrial fluids by tenacity), this conclusion cannot be admitted."

There is one case only, as Euler at this point observes, in which the usual method can be applied to the whole surface of a body, without leading to the inadmissible conclusion of total resistance equal to zero, and this case occurs if we conceive a body of paraboloidal form, moving through the fluid along its main axis.

This is quite evident as a paraboloid does not possess a rear part, and indeed this case would have called for very few words, if Euler with his mathematical genius had not dedicated some pages of geometrical considerations and calculations to it. At the conclusion of his analysis, however, he admits that a paraboloid, on account of its being only a mathematical figure without actual existence, cannot be considered an exception to the general rule, that the Newtonian or usual method for calculating resistance cannot be extended to the whole surface of real bodies, but has to be limited to their frontal part only.

Later, in the last period of his life, Euler came back again to the question of fluid resistance in a paper read on November 24th, 1781 before the "Académie Royale des Sciences de Paris" and entitled "Essai d'une théorie de la résistance qu'éprouve la proue d'un vaisseau dans son mouvement"[1].

In this paper Euler observed that the usual theory being based only on the effect of the shock of the water against the prow, although also the friction of the same on the ship's surface must produce a very

[1] "Histoire de l'Académie Royale des Sciences," vol. VIII, année 1778, à Paris, 1781. These volumes being published two or three years later, the paper of Euler not to delay its publication, was printed in the volume containing the papers read in the years 1778, and published in 1781.

remarkable effect, "it is no wonder if this rule gives always a resistance too small and if it the more diverges from truth, the more the obliquity of the shock is great".

Then Euler considering a prow of suitable form for ease of calculation, writes a formula in which, to the expression of resistance, according to the usual theory, is added an expression which had been found by him for friction, in a paper published in the year 1761 and entitled "Tentamen theoriae de frictione fluidorum"[1].

This formula taking into consideration both the effects of shock and of friction on the prow will give according to Euler the measure of the total resistance of the same.

Now this formula was such that when assuming the angle of incidence equal to 90 deg. the total resistance became proportional to the sine square of the angle of incidence, as in the usual rule; while when assuming the angle of incidence very small, the total resistance becomes proportional to the simple sine: results, Euler concluded, which are in accord with those given by the experiments of Messrs D'Alembert, Marquis de Condorcet and the Abbé Bossut.

Euler terminated his paper emphasizing the fact that both the researches by the said experimenters and his own formula relate only to the resistance on the prow of a ship, while as regards resistance on the stern (where conditions, he thought, are quite different), although one might imagine the effect of water pressure and friction on the stern as a function of the velocity of motion "it seems however too difficult to determine this effect by any theory, so that I am satisfied enough to have established the resistance exercised by the water on the prow only".

After this detailed account of Euler's efforts on this much debated problem of fluid resistance, let us now mention, very briefly indeed, not to enter into a pure mathematical field, that other part of his work for which he deserved the highest appreciation in hydrodynamics, that is, his work of perfecting the equations of D'Alembert. He wrote on this subject these principal papers: "Principes généreaux du mouvement des fluides"[2], "Principia motus fluidorum"[3], "Von den Gesetzen der Bewegung flüssiger Materien"[4].

If we now compare the work of the three great founders of hydrodynamics after Newton, that is, Daniel Bernoulli, D'Alembert, and Euler, we shall see that the progress of science is indebted to the two former

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[1] Novi Comm. Acad. Scien. Imp. Petrop., vol. VI, anno 1761.

[2] Histoire de l'Académie de Berlin, 1755.

[3] Novi Comm. Acad. Scien. Imp. Petrop. XIV, 7, 1769 (This paper bears, owing to a printer's error the date of 1759, so that it is always quoted as belonging to this year, while it was actually printed in the year 1769).

[4] Opera Postuma, Vol. 2, Petropoli, 1862.

for the physical formulation of principles, and to the latter for their mathematical development. D'Alembert in the introduction already mentioned to his book "Essai d'une nouvelle théorie de la résistance des fluides" (1752) explained with his usual clearness and insight the necessary dependence of calculation upon experiment, and tried to show how he himself only after having well established the hydrodynamic phenomena experimentally applied the rules of calculation to them. His words on this subject deserve to be quoted integrally:

"After having reflected for a long time on this important matter (the resistance of fluids) with all the attention of which I am capable, it has seemed to me that the little progress which has been made up to the present time is due to the fact that we have not yet understood the true principles according to which it must be dealt with. Therefore I thought to apply myself to seek these principles and the way of applying, if possible, calculation to them. For these two purposes must not be confused and modern Geometricians have not perhaps paid enough attention to this point. It is often the desire to be able to make use of methods of calculation which determine the choice of principles, whereas the principles themselves should first be sought without thinking in advance to bend them forcibly to methods of calculation.

Geometry which should only obey physics, when united to the latter, sometimes commands it. If it happens that a question which we wish to examine is too complicated to permit all its elements to enter into the analytical relation which we wish to set up, we separate the more inconvenient elements, we substitute for them other elements less troublesome, but also less real, and then we are surprised to arrive, notwithstanding our painful labor, at a result contradicted by nature; as if after having disguised it, cut it short, or mutilated it, a purely mechanical combination would give it back to us.

I have proposed to avoid this drawback in the work which I give today. I have sought the principles of the resistance of fluids as if analysis had not to enter therein, and only after having found these principles have I tried to apply analysis to them."

Quite different was the genius of Euler, who contributed so much to the development of the equations of hydrodynamics, but who was not equally successful in questions where, besides mathematical skill there was wanted experimental knowledge with physical insight, as in the question of fluid resistance.

In his obituary [1] there are found the following words:

"M. Euler seemed sometimes to be occupied by the pure pleasure of mathematical calculation and of considering the questions of mechanics or physics only as occasions for exercising his genius and for abandoning himself to his predominant passion. So scientists have reproached him for having sometimes lavished his calculus on physical hypotheses, or even on metaphysical principles, of which he had not sufficiently examined the likelihood and solidity."

As this characteristic of Euler has been in all times shared by a large number of mathematicians in hydrodynamics (as in other branches of mechanics) it has seemed that a somewhat detailed exposition

[1] Eloge de M. Euler, Histoire de l'Acad. de Sciences de Paris, année 1783, published in the year 1786.

of this less important part of the work of such a great mathematical genius might be of interest.

Euler's work of perfecting the equations of hydrodynamics was continued by Lagrange (1736—1813). Lagrange's researches on this subject after a first indication dating from 1760, are found essentially in his "Mémoire sur la théorie du mouvements des fluides" [1] and in his "Mécanique Analytique" (1788).

At the present time the equations of hydrodynamics have two forms, respectively known under the name of the equations of Euler and of Lagrange, although both can be traced back to Euler, and both were set forth in the "Mécanique Analytique" of Lagrange. The former type of equation was called by Maxwell the equations of the *statistic* method, the latter of the *historic* method. The second form, that is the so-called equations of Lagrange, was given by Euler exactly in his paper of the year 1769.

With these names of major importance, those of other investigators belonging to this period cannot be passed over in silence: Dubuat, Borda, Avanzini, Robins, Hutton, Vince, Robinson.

Dubuat, the author of "Principes d'hydrodynamique et d'hydraulique vérifiés par un grand nombre d'expériences faites par ordre du gouvernement" (1779, first edition, 1816, last edition) is well known on account of the famous paradox which bears his name, in which he experimentally found that the resistance opposed by a motionless plane to a liquid in motion is greater than that met by the moving plate in a motionless liquid, and closely in the ratio 1.3:1 which difference is attributed generally to the lack of uniformity of the stream in the former case. Avanzini is well known for having in a first paper of his entitled "Nuove ricerche dirette a rettificare la teoria della resistenza dei fluidi e le sue applicazioni" (1804) demonstrated by experiments that when a plane surface moves obliquely in a fluid its center of pressure displaces itself forward of the center of area.

Borda, Hutton and Robins in the study of questions of ballistics carried out experimental researches on the resistance of air by means of whirling arms, the invention of which goes back to Robins ("New Principles of Gunnery", 1743).

The experiments of Borda were described in the "Mémoires de l'Académie des Sciences" (1763), those of Hutton, carried out in the years 1787, 88, 89 and 91, were described in the "Philosophical Transactions" (1798). Hutton employed in these researches a whirling arm, like that with which Robins demonstrated before the Royal Society of London that the resistance is almost proportional to the square of the velocity. Hutton concluded the account of his experiments by stating that the

[1] Nouv. Mém. de l'Acad. de Berlin 1781.

resistance of the air found by him with his experiments differs remarkably from that calculated by means of the theories up to that time admitted, and that the results deduced from these theories are completely wrong, so that only by means of a series of accurate and well executed experiments will it be possible to establish an exact theory in this matter.

Borda carried out experiments also in water [1] and similarly Vince [2], who is well known for having demonstrated the discrepancy between the experimental and theoretical results obtained by applying the sine square law. Robinson finally in the year 1822 in his "System of Mechanical Philosophy", vol. II, affirmed the necessity of applying, instead of this law, that of the simple sine.

These experiments were followed in the year 1829 by those carried out by Duchemin. They were described by him many years later in a book entitled "Recherches Expérimentales sur les Lois de la Résistance des Fluides" (Paris 1842). In this book he proposed to present laws and formulas deduced from experience and as such, to be useful in practice.

In his formulas Duchemin ignored the effect of viscosity, but took into account the effect of friction, basing his work on the experimental results of Coulomb, who in a series of experiments on the torsion of thin wires in very delicate dynamometers, studied the influence of air resistance. The researches of Coulomb were described by him in two papers, the first published in the year 1784 "Mémoires relatifs à la physique publiés par la Société Française de Physique", and the second, more perfected, published in the year 1801 under the title of "Expériences destinées à déterminer la cohérence des fluides et les lois de leur résistance dans les mouvements très lents". The conclusions set forth by Duchemin in his paper are the following:

(1) that in the resistance of a fluid two factors appear, the one proportional to the velocity and dependent on the viscosity (cohérence), and the other proportional to the square of the velocity and dependent on the inertia and on the friction (adhérence);

(2) that the term dependent on the viscosity could be ignored as regards the term dependent on the other two magnitudes, when the velocity exceeded 2 or 3 centimetres per second;

(3) that the term dependent on the viscosity remains inferior to the other for velocities over 7 centimetres per second, while for velocities less than 7 centimetres per second, the influence of viscosity preponderated in comparison with that of inertia and of friction.

The preponderating influence of the viscosity of air for very slow velocities had been noticed also by Dubuat in his "Principles" of the

[1] Mémoires de l'Académie des Sciences, 1767.
[2] Philosophical Transactions, 1798.

year 1779 already mentioned, while studying the oscillations of a pendulum in a fluid medium.

And it seems that M. Navier (1785—1836) was influenced by the considerations of Dubuat in writing, in the year 1826, a paper in which he was the first to give the equations of motion without assuming the equality of the pressures in all directions. This paper was entitled "Mémoire sur les lois du mouvement des fluides"[1]. Navier considered the case of an homogeneous incompressible fluid by supposing it to consist of molecules animated by repulsive forces. These forces were simply a function of the distance separating the molecules increasing as this distance decreased and diminishing as the distance increased.

The same subject of the equations of motion in a viscous fluid was dealt with, but on different hypotheses, by Poisson (1781—1846) in the year 1831 in a paper entitled "Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides"[2], arriving at equations which in the case of an incompressible fluid, reduce themselves to those of Navier.

Later, in the year 1843, the problem was considered from a different point of view by Barré de Saint Venant (1797—1886) in a paper entitled "Note à joindre au Mémoire sur la dynamique des fluides"[3].

Barré de Saint Venant found himself in agreement with Navier regarding the equations for an incompressible fluid and certain other hypotheses as well. In the year 1847 the equations of the motion of viscous fluids were given by Stokes (1819—1903) in a paper entitled "On the Theories of the Internal Friction of Fluids in Motion"[4].

In this paper Stokes applied to viscous fluids that pure mathematical analysis which thirteen years later was applied by Helmholtz to the perfect fluid, by decomposing the most general motion of an element of fluid into three components, that is, one of pure translation, one of pure rotation, and one of pure strain.

However, it is necessary to notice that the rotation concept goes back to Cauchy (1789—1857), who in a paper entitled "Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie" presented to the Académie des sciences de Paris in the year 1815 and published in the year 1825 (Recueil des Savants étrangers) introduced the *average rotation at a point*; which average rotation—Appel observed in his "Mécanique rationelle" (vol. III)—is somewhat the same as the instantaneous rotation of the fluid particles surrounding that point, and was then considered by Helmholtz under the name of vortex.

[1] Mémoires de l'Académie des Sciences, vol. VI, 1826.
[2] Journal de l'Ecole Polytechnique, cahier XX 1831.
[3] Comptes Rendus, vol. XVII, p. 1240, 1843.
[4] Transactions of the Cambridge Philosophical Society, vol. VIII, pt. III.

A second paper by Stokes' on viscosity appeared in the year 1856 under the title "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums"[1] in which, mentioning those who had preceded him in the study of the influence of the resistance of the air on the oscillations of the pendulum, he quoted the studies of Bessel[2], Poisson[3] and Plana[4].

Beyond these papers of Stokes referring to viscous fluids, two other papers of his are worthy of mention, containing as they do, the first enunciation of ideas which later acquired a very great development.

The first of these papers belongs to the year 1843 and was entitled "On Some Cases of Fluid Motion"[5]. In this paper Stokes noted the fact that motion in certain circumstances "is unstable so that the slightest cause produces a disturbance in the fluid which accumulates as the solid moves on, till the motion is quite changed".

Osborne Reynolds (1842—1912) gave much study to this question of the changing form of motion as early as 1876, and set down the results in a paper published in the year 1883 under the title "An Experimental Investigation of the Circumstances which Determine Whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels"[6]. On the same subject he read the next year a paper of a popular character entitled "On the Two Manners of Motion of Water"[7].

Osborne Reynolds carried out his experiments by comparing the results obtained by him with the preceding ones of Jean Poiseuille (1799—1869) on motions of liquids in tubes of very small diameter and in capillaries, described in two papers, both entitled "Recherches expérimentales sur le mouvement des liquides dans les tubes de très petits diamètres", of which the first was read by Poiseuille in three seances of the Académie des Sciences during the years 1840 and 1841[8], and the second of greater extension was presented by him to the same Academy in the year 1842, which was published four years later[9].

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[1] Transactions of the Cambridge Philosophical Society, vol. XI, p. 8.

[2] "Über die Unrichtigkeit der bisher für Pendelversuche angewandten Reduktionen auf den luftleeren Raum", Astron. Nachrichten. Vol. VI, 1827.

[3] "Sur les mouvements simultanés d'un pendule et de l'air environnant", Mém. de l'Academie de l'Institut, vol. XI, p. 521, 1832.

[4] "Mémoire sur le mouvement d'un pendule dans un milieu résistant", Memorie dell'Accademia di Torino, vol. XXXVII, p. 209, 1835.

[5] Transactions of the Cambridge Philosophical Society, vol. VIII.

[6] The Philosophical Transactions of the Royal Society, 1889.

[7] Proceedings of the Royal Institution of Great Britain, 1884.

[8] Comptes Rendus, vol. 11, p. 961 and p. 1041, 1840; Comptes Rendus, vol. 12, p. 112, 1841.

[9] Mém. sav. étrang. vol. IX, p. 433, 1846.

However, prior to Poiseuille, as L. Prandtl points out in his lessons on Hydro and Aeromechanics recently published[1], the flow in cylindrical tubes of small diameters had been already studied by C. Hagen (1710—1769) in a paper published in the year 1839[2].

The result of Reynolds' researches was that the motion of liquids takes place under two forms, that is, of *laminar* flow or *turbulent* flow, and the passage from one form to the other occurs abruptly, and that determining factors of both conditions are the viscosity of the liquid, the velocity and width of the current, concluding that "the effect of these influences is subject to one perfectly defined law, which is that a particular evolution becomes unstable for a definite value of the viscosity divided by the product of the velocity and space". This is, as is apparent, the well known Reynolds Number, and the law so expressed is today often known by the name of the Reynolds law of similitude.

Stokes' other paper belongs to the year 1847 and is entitled "On the Theory of Oscillatory Waves"[3] and therein is found the first statement of the possibility of the existence in perfect fluids of surfaces of discontinuity of the velocity, a subject which was developed magnificently twenty-one years later by Helmholtz.

With Helmholtz (1821—1894) hydrodynamics accomplished the most notable progress after D'Alembert, Euler and Lagrange. Like these, Helmholtz also investigated motion in perfect fluids, but his researches differ from the preceding ones for having revealed phenomena up to that time unknown. These researches were set down by Helmholtz in two principal papers, the one of the year 1858 "On the Integrals of the Hydrodynamical Equations Corresponding to Vortex Motions"[4] the other of the year 1868 "On the Discontinuous Motions of a Fluid"[5].

In the first paper Helmholtz begins by observing that in the problems of hydrodynamics investigated up to that time, the hypothesis always had been made that the velocity components along the three axes of any particle of the fluid could be put equal to the derivatives of a determined function, to which he himself proposed to give the name of the *potential of velocity*, in analogy with the name of *potential* which had been introduced into Mechanics in the year 1840 by Gauss.

The validity of this hypothesis—Helmholtz himself observed—had been already demonstrated by Lagrange in his "Mécanique Analitique" for all cases in which the motion of the fluid was effected by the action of forces derived also from a potential of the forces. And from the fact

[1] "Hydro- und Aeromechanik nach Vorlesungen von L. Prandtl" von O. Tietjens, vol. II, p. 16, Berlin 1931.

[2] "Über die Bewegung des Wassers in engen zylindrischen Röhren". Pogg. Ann. vol. 46, p. 423, 1839.

[3] Transactions of the Cambridge Philosophical Society, vol. VIII.

[4] Crelles Journal für die reine und angewandte Mathematik, Bd. LV, pp. 25—55.

[5] Monatsberichte d. Kön. Akad. d. Wiss. zu Berlin, pp. 215—228.

that the majority of the natural forces which can be defined mathematically can be represented in this way, the consequence could be deduced that also the majority of fluid motions were provided with a potential of the velocity.

However—Helmholtz continued—Euler, in his paper "Principes généraux du mouvements des fluides" of the year 1755 had already observed that there may be cases in which no potential of the velocity exists. Among the forces causing these motions there is the frictional force of the fluid particles against each other or of the fluid against a solid wall. The effect of friction is remarkable indeed and is responsible for the great differences arising between theory and reality. The difficulty of defining this effect and of finding the necessary means to measure it, consists for the most part in the lack of any idea of the form of motion caused in fluids by friction, and therefore this is the reason—Helmholtz concluded:

"why a research of the forms of motion in which no potential of the velocity exists seems to me of great interest. This research leads to the result that in the cases in which a potential of the velocity exists the smallest fluid particles do not possess rotatory motions, whereas when no such potential exists, at least a portion of these particles, is found in rotatory motion".

This result reached by Helmholtz extended remarkably the knowledge of the nature of fluid motion, but this knowledge was still further promoted by the general properties which Helmholtz discovered in those special movements without a potential of the velocity, which he named "vortex motions".

With this paper Helmholtz opened, moreover, a new field of research carried on by many authors and resulting in a very copious literatur. Among the most important papers are to be mentioned those of Hankel, "Zur allgemeinen Theorie der Bewegung der Flüssigkeiten" [1] of W. Thompson (Lord Kelvin) "On Vortex Motion" [2] and of E. Beltrami "Principi di aerodinamica razionale" [3].

In his second paper Helmholtz starts from the observation that water when issuing from a sharp edge tube, placed in a vessel filled up by water, does not spread in all directions as it could do according to the theory in analogy with the electric and magnetic flow, but rather forms a jet which only after a certain length fuses itself with the surrounding water.

The same jet is observed in air when the latter, being mixed with smoke to make the phenomenon visible, issues from an orifice. Now this formation of a jet is necessarily connected with the formation of a surface of discontinuity of the velocity.

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[1] Göttingen, 1861.

[2] Transactions of the Royal Society of Edinborough, vol. XXV, 1869.

[3] Bologna, Acc. Scienze, Mem. 1871—1877.

But such discontinuities of the velocity, up to that time, had not been considered by the theory. In fact, Helmholtz observed—in the hydrodynamic equations the velocity and pressure of the fluid particles are always dealt with as being continuous functions of the coordinates, while in the nature of an inviscid fluid there is nothing which forbids us to conceive two adjacent layers, one slipping on the other with a finite velocity. At least those properties of the fluids which are taken into consideration in the hydrodynamic equations, that is, the constancy of the mass in each special element and the equality of pressures in all directions, do not present any obstacle to the conception that on either side of a surface inside a fluid mass, there may be tangential velocities having with regard to each other a finite difference.

It is taken for granted however—Helmholtz concluded—"that the components of the velocities and of the pressure directed perpendicularly to this surface on the two parts of the same have to be equal".

From the idea of the Helmholtz surface of discontinuity, Lord Rayleigh (1842—1919), in a paper of the year 1876, entitled "On the Resistance of Fluids"[1] deduced an explanation of the resistance met by a plane submerged in a current, resistance which was denied by the classical hydrodynamic theory in opposition to the Newtonian one and to experience.

"There is no part of hydrodynamics"—began Lord Rayleigh in his paper—"more perplexing to the student than that which treats of the resistance of fluids. According to one school of writers a body exposed to a stream of perfect fluid would experience no resultant force at all, any augmentation of pressure on its face due to the stream being compensated by equal and opposite pressures on its rear. And indeed it is a rigorous consequence of the usual hypotheses of perfect fluidity and of the continuity of the motion, that the resultant of the fluid pressures reduces to a 'couple' tending to turn the broader face of the body toward the stream. On the other hand it is well known that in pratice an obstacle does experience a force tending to carry it down stream, and of magnitude too great to be the direct effect of friction.

It was Helmholtz who first pointed out that there is nothing in the nature of a perfect fluid to forbid a finite slipping between two contiguous layers, and that the possibility of such an occurrence is not taken into account in the common mathematical theory, which makes the fluid flow according to the same laws as determine the motion of electricity in uniform conductors. Moreover, the electrical law of flow would make the velocity infinite at every sharp edge encountered by the fluid; and this would require a negative pressure of infinite magnitude."

It is therefore necessary—concluded Lord Rayleigh—to come to the idea of Helmholtz and admit that at the edges of a lamina submerged in a current, a surface of discontinuity is formed which bounds the fluid behind the lamina. Such a mass of fluid which extends to infinity is at rest and therefore at constant pressure. And as the surface of discontinuity must have on the two sides the same pressure, it follows

[1] Phil. Mag. ser. V. vol. II, pp. 430—441.

that the mass of the fluid at rest behind the lamina has the same pressure as the free current. On the front face surface of the lamina therefore—Rayleigh concluded—there is augmentation of pressure corresponding to the loss of velocity.

From the case of the perpendicular lamina Lord Rayleigh passed to the case of the oblique lamina observing that in such a case, in a quite different way than in the Newtonian hypothesis, resistance depended on the whole velocity of the stream and not only on its component perpendicular to the lamina.

In fact—he observed:

"behind the lamina (as behind the perpendicular one) there must be a region of dead water bounded by a surface of discontinuity, within which the pressure is the same as if there were no obstacle. On the front face of the lamina there must be an augmentation of pressure, vanishing at the edges and increasing inward to a maximum at the point where the stream divides. At this point the pressure is $(^1/_2)\,\varrho\, v^2$ corresponding to the loss of the whole velocity of the stream. It is true—he concluded—that the maximum pressure prevails over only an infinitely small fraction of the area; but the same may be said even when the incidence of the stream is perpendicular".

But, prior to Lord Rayleigh, Kirchhoff (1824—1887) starting from Helmholtz's paper had already investigated in a famous note of the year 1869 and entitled "Zur Theorie freier Flüssigkeitsstrahlen" [1], the pressure exercised on the unit of surface of a lamina exposed to a stream.

Lord Rayleigh declared in his paper that he arrived at his own results, being unaware of Kirchhoff's previous one; at any rate the latter, notwithstanding having considered both the cases of the perpendicular and of the oblique lamina under the hypothesis of the surface of discontinuity, did not calculate the force acting on the lamina in the second case, that is when the lamina is exposed to the stream obliquely, which, however, was done by Lord Rayleigh.

Kirchhoff's researches on this and other fields of hydrodynamics are of fundamental importance. In addition to the above noted paper, they are set forth in various other papers and in his "Vorlesungen über Mathematische Physik" of the year 1876.

Among his various papers mention may be made of the following: "Über die Kräfte, welche zwei unendlich dünne starre Ringe in einer Flüssigkeit scheinbar aufeinander ausüben können" [2], "Über die Bewegung eines Rotationskörpers in einer Flüssigkeit" [3] and "Über stehende Schwingungen einer schweren Flüssigkeit" [4].

Another line of research, which must be mentioned before closing this rapid review of the studies carried out in the field of classical

[1] Crelles Journal f. d. r. u. a. Math., vol. 70, p. 289, 1869.
[2] Crelle XXI of the year 1870.
[3] Crelle XXI of the same year 1870.
[4] Berlin Monatsbericht, of the year 1877.

hydrodynamics up to the end of the past century, is that followed by W. J. Macquorn Rankine (1820—1872), who is responsible for the main features of that theory known today as the source and sink theory for the drawing of streamlines.

The first theory on the forms of ships' water lines, practically useful and founded on mechanical principles, was set forth, according to Rankine, by Scott Russel in the first and second volume of the Transactions of the Institution of Naval Architects; but only by Rankine was the theory established in a rigorous mathematical form, in a paper published in the year 1864 and entitled "On Plane Water-Lines in Two Dimensions"[1]. For the water-line curves Rankine proposed the term, derived from the Greek, of "Neoids", that is, ship-shape curves.

To this paper of a mathematical character followed a short paper by Rankine in the year 1868, "intended to enable persons who have not mastered higher mathematics" to follow the subject, and entitled "Elementary Demonstrations of Principles Relating to Streamlines"[2].

The researches thus far noted on the resistance of fluids, as we have seen, were all determined by varied problems, naval, hydraulic, and ballistic; but in no one of these researches is there found any application to the problem of flight, a problem which in general did not occupy scientists previous to our own days. In fact, from Leonardo da Vinci about 1500, we must come down to Sir George Cayley, in the first years of the past century, to find this problem resumed in a scientific study, passing over without notice the more or less fantastic projects for flight, of which in all times there are found numerous examples.

Sir George Cayley (1773—1857) in his "Aerial Navigation"[3] established the problem of mechanical flight in these very sound terms: "to make a surface support a given weight by the application of power to the resistance of air", having determined this resistance by means of the available data of Robins and other experimenters, which he verified with his own experiments. In particular he asserted that for calculating the resistance of an inclined plane he renounced the use of the sine square law, as the experiments carried out by the French Academy had shown that this law did not hold good for small angles, for which the simple sine law is more exact.

Cayley effected the decomposition of the action of the air on an inclined plane into two components, the one perpendicular to the motion and constituting the lift, and the other parallel to the motion, constituting the resistance, which must be overcome by the action of some form of propeller, thus arriving at the essential features of the airplane.

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[1] Philosophical Transactions for 1864.
[2] Engineer of Oct. 1, 1868.
[3] Nicholsen's Journal 1809—10.

But time was not yet ripe for the airplane and Cayley himself abandoned this line of research to pass to the study of lighter-than-air craft.

Only after half a century were these researches resumed by F. H. Wenham, who in the year 1866 published a paper entitled "Aerial Locomotion"[1] Wenham—which, as Lord Rayleigh said, had the merit of having recognized the principle that the pressure exercised on a lamina exposed perpendicular to a stream can be increased indefinitely by giving to it at the same time a sufficiently high velocity along its own plane; that is, by causing the lamina to move with a small incidence, as regards the direction of relative velocity.

Such a principle arises at any rate from the adoption of the simple sine law instead of the sine square law, as was shown in the year 1871 by Froude for resistance in water, in a discussion on a paper by Sir F. Knowles on propulsive screws[2].

A series of experiments in the meantime began in Paris, on the resistance of air, at the "Société de Navigation Aérienne". Among these investigators was Alphonse Pénaud whose experiments from 1871 to 1878 are described by him in a paper of the year 1878 entitled "Recherches sur la résistance des fluides"[3].

In the year 1881 Louis Mouillard published in France his famous book "L'Empire de l'Air" containing a very clear conception of the action of air in flying, and in the year 1885, in England, Horatio Philips utilized for experiments on models the first aerodynamic tunnel of which we know the existence with certainty. Its description is found in "Engineering" of the same year.

In the year 1890, in Germany, Otto Lilienthal published his book "Der Vogelflug als Grundlage der Fliegekunst" describing the results of remarkable experiments carried out since the year 1871, first with whirling arms and then with natural wind, and giving to the tested surface various shapes, curvatures and thicknesses.

In the year 1891 Langley in America published his very important "Experiments in Aerodynamics"[4] describing his researches on air resistance carried out since 1880. These experiments since 1887 had been executed in the Aerodynamic Laboratory founded by Langley himself at Allegheny. In the meantime W. H. Dines carried out not less accurate experiments in England, describing them in a paper of the year 1891, entitled "On Wind Pressure upon an Inclined Surface"[5].

Other researches on the resistance of air were carried out in France by Charles Renard since 1888, in Austria by Wellner about 1898, in

[1] Ann. Report. Aer. Soc., 1866, also Aeronautical Classics, London, 1900.
[2] Proc. Inst. Civ. Eng., vol. 32.
[3] Bulletin de la Société Philomatique de Paris, 1878.
[4] Smithsonian Contributions to Knowledge.
[5] Proc. Roy. Soc. vol. 48, 1891.

Italy by Aristide Faccioli, who described them in his book "Teoria del volo" (1895) in Germany by W. von Lössl, describing them in his work "Der Luftwiderstand" (1895) and by others.

Finally about 1900 the Wright brothers carried out remarkable experiments on the resistance of air, thus arriving at the determination of the shape of their first biplane.

It is well known indeed that these two great American flyers united a high scientific spirit to great determination and force of character.

CHAPTER III.

PERIOD OF MODERN AERODYNAMICS: FROM THE END OF THE XIX CENTURY ONWARD

Of the two aspects in which air reaction appears, known today as *lift* and *drag*, we have noted in the preceding section researches on the second one only.

More exactly we have seen that this resistance had a finite value in the case of the discontinuous fluid of Newton, whereas it had a value equal to zero in the case of the fluid of classic hydrodynamics, while it resumed a finite value with the hypothesis of Helmholtz's surface of discontinuity.

As regards researches on lift we have thus far noted only the ancient views of Leonardo da Vinci, who attempted a mechanical explanation of the sustentation of birds, basing it on the condensability of air.

Another analogous attempt can be found in the writings of Giovanni Alfonso Borelli (1608—1679) who, in his book "De motu animalium", published posthumously in the year 1680—1681, explained mechanically the sustentation of birds in flight, as an effect of the reaction of air due to the elasticity and the internal friction of its particles. It is to be noted that at that time and up to the early nineteenth century, many strange hypotheses regarding the flight of birds were current, ascribing the lift to mysterious psychic forces or occult powers possessed by them.

Borelli's observations on natural flight became indeed a subject of study and comment as soon as the interest in aviation arose. They were particularly commented and mentioned by the Duke of Argyle, the first president of the Royal Aeronautical Society, in his book "The Reign of Law" published in 1866, the year of the Society's foundation; also by I. Bell Pettigrew, in his "Animal Locomotion" (four editions between 1874 and 1891, one in French in the year 1874 and one in German in the year 1875), and by the famous physiologist and pioneer of aviation, E. I. Marey in his works "La machine animale" (1873) and "Le vol des oiseaux" (1890). Rather the latter stated that Borelli's results had anticipated by two centuries his own. Those of Borelli's researches of which Marey spoke, related to wing manoeuvres of birds, a matter of

great interest in the last century, because in them it was hoped to be able to "surprise the secret of flight" whereas, regarding the physical cause of sustentation in flight, likewise touched on by Borelli, this was dealt with, together with the problem of the required power for flight, by making use of elementary formulae founded on the basic laws of mechanics, and following the same proceeding applied by Rankine in his theory of propulsion.

This theory had for its basis the proposition that in every second a certain mass of fluid is operated upon by a propulsive mechanism imparting to it an added velocity. The velocity being communicated to the mass in a rearward direction, the force of propulsion resulted forward. In applying the same reasoning to the support of a load, the mass of air was assumed to be projected downward and therefore the corresponding force resulted upward, supporting the load.

The proceeding, in short, was that of equating sustentation or thrust to the momentum generated downward or backward respectively.

If v is the velocity imparted to the fluid and knowing the mass m dealt with, there follows, $W = m\,v$,

while for the energy expended per second,

$$E = \frac{1}{2}\,m\,v^2 = \frac{W^2}{2\,m}$$

Whence it appeared that the power expended to obtain any given thrust or for any given weight W to be sustained, was in the inverse ratio to the mass of the fluid dealt with per second.

The problem was then to determine m, but here no other method was supplied by the theory except that founded on the consideration of the fictitious medium of Newton, leading to the sine square formula. In fact if A is the area of a wing travelling at a velocity V, α the angle of incidence, and ϱ the air density, the theory gave

$$m = \varrho\, V\, A\, sin\,\alpha$$

Then, according to the hypothesis on the one hand of an elastic medium or on the other, of a non-elastic medium, there followed,

$v = 2\,V sin\,\alpha$ or $v = V sin\,\alpha$, from which respectively:

$W = 2\,\varrho A\;V^2 sin^2\alpha$ or $W = \varrho A\;V^2 sin^2\alpha$;

formulae leading to values of lift much inferior to reality.

In these circumstances, there remained nothing to be done but to put into the equation $W = m\,v$, some coefficient empirically derived.

But "at this point Lanchester came forward with his remarkable physical insight"[1]. Lanchester (born in the year 1868) who, in his Wilbur Wright Lecture before the Royal Aeronautical Society in 1926 reviewed his earlier work in this field, dating back to 1891. With some

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[1] Durand, W. F., "Historical Sketch of the Development of Aerodynamic Theory", Paper presented at the International Civil Aeronautical Conference, Washington, D. C., December 12, and 14, 1928.

paraphrasing and abbreviation from the report of this lecture[1] and with some references to his textbook on Aerodynamics[2] the general basis of his earlier reasoning may be stated as follows:

Let us imagine first, he said, a plane moving with any angle of incidence through the discontinuous medium of Newton. It will fling a number of the particles of the medium downwards thence deriving sustentation. At the same time the mean density of the medium will no longer be uniform, it will be increased in regions below the track of the plane. But in a medium possessed of continuity such as air, this cannot happen. There may be a slight temporary compression of the air below and rarefaction above, whilst the plane is passing, but there can be no permanent change of density or accumulation of matter in the lower strata of the atmosphere. And this, since the local down-current caused by the plane in its passage must find its counterpart in an up-current of equal displacement elsewhere.

How this can happen, Lanchester continued, it does not seem difficult at first sight to imagine. It occurs to the mind at once indeed, that the down-current produced by the direct action of the plane or aerofoil is associated with up-currents generated beyond the wing tips.

A circulation of air will then take place around the edge of the plane from the under to the upper side, forming a kind of "vortex fringe". But this is not the whole story. Whatever tendency there may be to develop these up-currents beyond the wing tips, such tendency must also exist in front and at the rear of the aerofoil. Rather in the case of an aerofoil of high aspect ratio it is far more powerful in those regions which provide shorter cuts from the pressure region below to the rarefaction region above.

Rather, in order the better to investigate these vertical currents caused by the leading and trailing edges of the wing, Lanchester eliminated the motion at the two tips, assuming an aerofoil of infinite lateral extension: a further advantage of such being to reduce the problem to two-dimensional motion.

He then took into consideration a horizontal plane of infinite aspect ratio, supporting a load without horizontal motion, that is, a two-dimensional parachute, and considered that as the plane descended, the air dragged down was compensated by an up-current around its edges, producing a circulation of air fed by the difference of pressure between the under and upper surfaces.

We are with this, he said, in the presence of a field of force established around the plane when the load was first applied: a field of force everywhere defined by the acceleration of the air particles.

[1] The Journal of the Royal Aeronautical Society, No. 190, vol. XXX, October, 1926.

[2] LANCHESTER, F. W., "Aerodynamics", London 1907.

Lanchester then assumed a horizontal motion imparted to this plane descending vertically: the two-dimensional parachute thus became a two-dimensional glider. By substituting for the force of gravity applied to the plane, the resultant of the force of gravity and an applied force of propulsion, we can also say, according to Lanchester, that we have passed from the case of a plane progressing through the air perpendicularly to the direction of its motion, to the case of an inclined plane moving horizontally. But let us return to the more evident case of the *glider*.

Lanchester noted that the fluid particles which are gradually influenced by the plane while passing through the field of force established around it,

"will receive an upward acceleration as they approach the aerofoil, and will have an upward velocity as they encounter its leading edge. While passing instead under or over the aerofoil, the field of force is in the opposite direction, viz., downward, and thus the upward motion is converted into a downward motion. Then, after the passage of the aerofoil, the air is again in an upwardly directed field, and the downward velocity imparted by the aerofoil is absorbed".

Now, since it is impossible for the residuary air to have downward velocity, on account of the fact that this would result in an accumulation or increase of density of the air below the level of the aerofoil, so, according to Lanchester, it was necessary that the last state of rest should be the same as its first state, *viz.*, one of no motion.

From this Lanchester deduced the conclusion that the motion imparted to the air by the plane was given back by the air both in respect of its vertical and horizontal components, and consequently there was no continual transmission of energy to the air, and no work was required to maintain the motion or to support the plane.

In other words, for an aerofoil of infinite span or a finite aerofoil, travelling between two vertical walls or boundaries, the expenditure of energy (ignoring skin friction) was equal to nought.

This system of flow was consequently classified by Lanchester as a *conservative system*, the energy of the fluid motion being carried along and conserved, the same as in the case of *wave motion*. The motion round about the plane was thus considered by Lanchester as a *supporting wave*.

Considered in the light of wave motion the system of the motion around the infinite plane (or peripteroid system[1] as Lanchester named

[1] The term *periptery* was suggested and used by Lanchester to indicate the region surrounding the wing, with the expressions derived from it of *peripteral theory*, *peripteral area*, and *peripteral motion*. From the expression peripteral motion, used to indicate the movement around the finite wing in the actual fluid, Lanchester distinguished the expression *peripteroid motion* which he defined in the Glossary annexed to his "Aerodynamics", as "the form of flow proper to the inviscid fluid in a doubly connected region, resulting from the superposition of a cyclic motion on one of translation".

it with an expression taken from the Greek) must be regarded as a *forced wave*, the aerofoil supplying a force acting from without.

From these considerations Lanchester deduced the conclusion that that wing section ought to be considered aerodynamically the best which was best suited to promote this form of motion, that is, to receive suitably a current of air in upward motion and impart to it a downward velocity, the air as a whole being considered as possessing relatively to the aerofoil a superposed motion of translation.

This excellent sectional form could be only a slightly curved one, of which the leading and trailing edges are conformable to the lines of flow, the essential feature evidently being that neither edge should give rise to a surface of discontinuity.

Experiments carried out by him at that time, 1891—1892, with aerofoils of curved sectional form, and the observation that the dipping front edge or arched section is exactly a characteristic of the wings of all birds capable of sustained flight, confirmed to him the advantages of the form thus evolved purely from theoretical considerations while studying the case of an aeroplane of infinite lateral breadth.

Horatio Frederick Phillips in England, however, had previously discovered experimentally the good aerodynamical features of the *dipping edge*, taking out two patents for this invention, one in the year 1884 (No. 13768) and another in 1891 (No. 13311), but Lanchester assures us that he became aware of the wing forms proposed by Phillips only after he had arrived at them by himself.

Later, independently of Phillips, the same results were found again by Otto Lilienthal with his experiments between 1890 and 1894; and of them also Lanchester became aware after he had developed his own theory.

The preceding considerations were summed up by Lanchester in a paper read on June 19, 1894, before the Birmingham Natural History and Philosophical Society in which, by exhibiting glider models constructed by him, he dealt essentially with the question of stability, in accordance with the title of the paper — "Stability of an Aerodrome".

It should be borne in mind that the term *aerodrome* at that epoch had the meaning which today attaches to *aeroplane*, whereas, the term aeroplane carried then rather the present meaning of *aerofoil*.

We must also remember that at that time—almost forty years ago—owing to the disagreement between the results of the mathematical theory of the perfect fluid and the actual motions experimentally ascertained for any real fluid, as Lanchester said, "those concerned with problems in real hydrodynamics, or aerodynamics, which is the same thing, were not unnaturally slow to seek assistance from hydrodynamic textbooks such as Lamb or Basset . . . . ."

Now Lanchester in those early days, was precisely among those who, while occupying themselves with real hydrodynamics, were not acquainted with the theories of classical hydrodynamics. Therefore he was not able for the moment to identify his "*forced wave*" with the cyclic-translation system of this theory; which identity was recognized by him some time after. Notwithstanding this recognition, however, Lanchester never rejected his first conception of the forced wave, regarding which he has recently stated that "this is a perfectly legitimate view".

Lanchester at that time did not publish his theory as summed up in the Birmingham Society Lecture, for he expected to set it forth before the Royal Society. In fact he revised and "tidied" up his paper, connecting among other matters the circulation theory of lift with the results found by various English artillerists who, since 1761 had been investigating the flight of projectiles and the results of imparting a spin to shells and bullets. Thus revised and extended, the paper was submitted to a member of the Royal Society, who, however, suggested that the Physical Society would be more suitable, and Lanchester sent his paper to that body, but owing "perhaps to an unfortunate selection of the readers to whom it was submitted" as Lanchester said, it was rejected. In the meantime twelve months or more had elapsed, up to September 3, 1897 (date of rejection). Lanchester was much disappointed by this fact and for the moment made no further effort toward publication. However he touched on his conception of the forced wave in a patent (No. 3608) taken out in the same year, 1897.

In the patent specification, an aeroplane was described as having at the extremities of the aerofoil two "*capping planes*", intended to simulate as far as possible the condition of an aerofoil operating in a region bounded by parallel walls, as in the ideal case of two-dimensional motion.

The use of these capping planes was described and figured "in order to minimize the lateral dissipation of the supporting wave".

A complete exposition of this theory was published by Lanchester only ten years later, by dedicating to it a chapter, the fourth one, of a complete treatise on flight.

The book in which Lanchester set forth among other matters the results of his early investigations, but now systematizing them in the frame of classical hydrodynamics, formed the first volume of this treatise [1].

Lanchester reproduced in Fig. 68 of this book, representing an arched plate placed transversely in a field of flow, a wall diagram of his paper of the year 1894 and from Fig. 61 to 67 seven illustrations of his paper offered to the Physical Society in the year 1897, and which were already shown in 1894.

[1] "Aerodynamics", London, 1907.

To the first edition of the "Aerodynamics" three more followed without alteration respectively in the years 1909, 1911 and 1918. The German edition was published in the year 1909 and the French in the year 1914.

The "Aerodynamics" is wanting in mathematical developments—a characteristic of all of Lanchester's writings. They are indeed "in plain English divested of all mathematical ornament" as their Author said in the "Wilbur Wright Lecture" previously referred to. However, this does not mean that his writings are easy reading; rather on the contrary. In fact it was by alleging the difficulty of understanding them on the part of those who were not already familiar with this line of thought, that L. Prandtl attempted to exonerate English scientists from the reproach of their countrymen of not having paid attention to Lanchester's ideas when they were first presented, leaving them to be developed by German scientists. Thus, "Lanchester's treatment is difficult to follow, since it makes a very great demand on the reader's intuitive perceptions, and only because we had been working on similar lines, were we able to grasp Lanchester's meaning at once" [1].

In the "Aerodynamics" there is set forth also the theory of the motion around a wing of finite span, which problem had already formed a subject of investigation by Lanchester in those early years between 1891 and 1894, and which had been similarly dealt with by him in his paper of the year 1897 offered to the Physical Society.

In this case it is obvious, as Lanchester said, that neither the lines of force of the field, nor the streamlines can be represented by a single section through the field. The hypothesis of the two vertical walls bounding laterally the aerofoil being abandoned, he understood that the lines of force being no more constrained to lie in parallel planes would diverge, some portion of them escaping, as it were, and passing around the tips of the aerofoil laterally.

The fluid traversing these lateral regions will then have upward momentum communicated to it during the whole time that it is in those regions, and will be finally left in a state of upward motion.

The fluid traversing instead, the middle region, crossed by the aerofoil, will receive, as in the preceding case, an upward acceleration before encountering the leading edge of the wing, a downward acceleration while passing under or over the aerofoil, and again an upward acceleration after the passing of the aerofoil. But here the upward and the downward momentum will no longer balance each other, as owing to the lateral spread of the ascending field forward of the aerofoil, the upward velocity communicated to the fluid before and after the passage of the wing is

[1] Prandtl, L., "Wilbur Wright Memorial Lecture", 1927. Journal of the Royal Aeronautical Society, No. 200, vol. XXXI, August, 1927.

less than the downward velocity imparted to it during the passage of the wing. Consequently the portion of the fluid traversing the middle region will be ultimately left with some residual downward momentum; while, as noted, the fluid passing laterally around the wing on both sides has received an upward momentum.

These two currents in opposite directions must quantitatively be equivalent, for otherwise there would be a continual accumulation, or else attenuation of the fluid in the lower strata of the atmosphere, which is impossible; but it is understood how the formation of these two currents requires a continuous expenditure of energy and "a source of power is consequently necessary to maintain the aerofoil in horizontal flight".

But this is not all. In fact in addition to these residual vertical motions of the air, of which the cause has just been discussed, there must also be horizontal counter-currents (formed simultaneously with those in a vertical direction) causing a circulation of air from the under side of the wing where is over pressure to the upper side where is under pressure. In short it is the *vortex fringe* touched on by Lanchester at the beginning, and then for the moment neglected in order to consider the problem with two-dimensional motion.

Now these two horizontal counter-currents combining with the residual vertical motions will give rise to two parallel cylindrical vortices, having right and left handed rotation respectively, which are being continually formed at the plane tips, and of which the energy is being continually dissipated in the wake of the advancing aerofoil.

It is true, Lanchester observed, that the conception of these vortex cylinders is not, for a perfect fluid, compatible with hydrodynamic theory, but in an actual fluid this objection has but little weight, owing to the influence of viscosity.

In conclusion, Lanchester demonstrated that in the wing of finite span together with the air circulation around the wing, to which the lift is due, two "*vortex trunks*" are established springing respectively from the right and left wing tips, rotating in opposite directions to each other, and to which a part of the drag is due: a drag independent of the friction of the wing surface, but connected with the phenomenon of lift. Lanchester terminated in his "Aerodynamics" his theory of the motion around the wing, connecting the difference between the two cases of the infinite and finite span with the connectivity of the space filled by the fluid.

In years subsequent to 1907, Lanchester perfected his theory of the wing of finite span, treating the same in a paper read in March 1915, under the title of "The Aerofoil in the Light of Theory and Experiments" [1].

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[1] "Proceedings of the Institution of Automobile Engineers", vol. IX, p. 169.

Unfortunately this publication was not widely circulated in continental Europe so that L. Prandtl in his paper of the year 1927, already mentioned, was able to assert that he had become aware of it only in the year 1926. Thus it happened that independently of each other and simultaneously both Lanchester and Prandtl, together with his collaborators, arrived at the same results.

Indeed it is interesting to note (as Prandtl stated in the same paper) that both in Lanchester's paper of the year 1915 and in a communication to the "Göttingen Institute" of the year 1914, there is found the same formula and an identical graph. This communication signed A. Betz is entitled "Untersuchungen von Tragflächen mit verwundenen und nach rückwärts gerichteten Enden"[1].

Concluding with regard to Lanchester's contribution to Aerodynamics, there are two great ideas conceived by him: the idea of circulation as the cause of lift, and the idea of tip vortices as the cause of that part of the drag, known today as the *induced drag*.

The former, which came late to the knowledge of the scientific world had been in the meantime found again and developed elsewhere, so that to its author was denied the glory of having directly influenced with it the progress of science; the latter, notwithstanding its delayed publication, coming as a new idea to the scientific world, serves to link its author's name to its discovery.

"The great merit of Lanchester consists in having illuminated the passage from the plane of infinite span which makes the field occupied by the fluid doubly connected to the finite plane making the field simply connected", had been written by N. Joukowski in a note of the year 1910 (to which later reference will be made) estimating with these words the contribution by Lanchester to the promotion of aerodynamics. At the same time however he limited Lanchester's contribution as follows: "But as regards magnitude and direction of the force of pressure on the body, in the peripteroid motion, Lanchester did not take them into consideration."

Even admitting with Joukowski that the problem had been essentially treated by Lanchester from the qualitative point of view, it is nevertheless true that even at the time of his first investigation a certain quantitative theory was built up by him in order to enable computations to be made of flight resistances and power expenditures for different values of aspect ratio.

"It was from this makeshift theory" as he said in his Wilbur Wright Memorial Lecture of the year 1926, "that my early experimental aerofoils were designed, and values were tabulated and finally published in 'Aero-

[1] "Zeitschrift für Flugtechnik und Motorluftschiffahrt", Jahrgang V, September 1914, Heft 16 and 17.

dynamics'. In spite of the paucity of experimental data, and the theoretical defects and apparent weakness in the theory, these tabulated results have turned out good forecasts." In fact an aerofoil calculated in such manner and used in his glider models and a number of years later (1912—1913) tested at Göttingen, gave a liftdrag ratio of 17, that is almost 10 per cent greater than any model tested up to that time.

This "Wilbur Wright Memorial Lecture" of the year 1926 already mentioned, in which Lanchester was invited by the Royal Aeronautical Society to set forth the modern theory of sustentation, indicating at the same time the lines along which it had been in particular developed by him, constitutes his most recent publication on this phase of the subject, "Sustentation in Flight" [1].

And now we pass from Great Britain to the continent to follow the movement of ideas which independently of Lanchester led to the same results as his in the determination of lift. To this purpose we must turn back to Lilienthal, as from Lilienthal's researches Kutta (born in the year 1867) started his well known paper of the year 1902 entitled "Lifting Forces in Flowing Fluids" [2] which represents in fact the first publication on the discovery of lift in the wing of infinite span.

Otto Lilienthal (1848—1896) saw the solution of the problem of mechanical flight in the design of a suitable form of wing offering a minimum of drag and a maximum of lift. Having stated, regarding a number of tests carried out with flat wings, that these did not answer his purpose, Lilienthal put to himself the following question: "Are there some surface forms which, moved as wings in a forward flight, give rise to a greater lifting and a less resisting action than flat wing forms used in the same conditions [3]?"

And his answer was that the most suitable form of aerofoil section was undoubtedly that of a lightly arched surface, according to the type of a bird's wing: "a form in the properties of which there lies, indeed with all probability, the whole secret of the art of flying".

In fact in his successive experimental work he showed that the reaction of air on a curved aerofoil section, with its concavity exposed to the air had a greater lifting component and a less resisting component than those of a flat wing section of the same dimensions, of the same angle of incidence, and of the same velocity.

Then comparing the comportment of a flat wing section with that of a curved wing section submerged in a current, he showed that the latter, while giving rise during the encounter with the flat lamina to vortices,

[1] Loc. Cit.

[2] "Auftriebskräfte in strömenden Flüssigkeiten", Illustrierte Aeronautische Mitteilungen, p. 133, July, 1902.

[3] Der Vogelflug als Grundlage der Fliegekunst, 22, 1890.

flowed instead smoothly past the curved lamina following completely its contour. From this he deduced that in the case of the flat lamina, a good deal of the fluid pressure ought to be dissipated in whirling motions, whereas in the case of the curved lamina, it was all employed in sustentation.

But, according to Lilienthal, this is not all. He observed indeed that in the fluid flowing past the curved lamina, there must develop, owing to the deviation of the current on account of the lamina, a centrifugal force which, in the layer of air in contact with the concave face of the lamina ought to cause an increase of the pressure of the air, and on the contrary, in the layer of air in contact with the convex face of the lamina ought to cause a suction, this double fact resulting in an increase of the lifting efficiency of curved wings. From this Lilienthal finally concluded that a wing section of suitable curvature, when exposed suitably to a current, would cause in the latter a wave, without whirling motions, which was reponsible for the sustentation of the wing; that is, there was produced a "*supporting air wave*" (tragende Luftwelle).

The curved wing surfaces investigated by Lilienthal, both with whirling arms and by exposing them to the natural wind, were laminae in the form of circular arcs, with a camber 1/12 of the chord and area 1/4 sq. m. They gave lift also at an angle of incidence equal to zero, and are represented in Fig. 33 of Lilienthal's book.

Kutta took into consideration these wing surfaces studied by Lilienthal by supposing them exposed to the wind with an angle of incidence equal to zero. He wrote their stream function and deduced from it the value of the velocity both on the upper and lower surface of the lamina. The value of the velocity on the upper surface appeared greater than on the lower, therefore the ultimate formula of the resultant pressure showed a thrust upward, perpendicular to the stream direction. The value of the thrust calculated through the following formula,

$$P = 4\,\pi\, a\, sin^2 \frac{\alpha}{2}\, \varrho\, V^2$$

(where $a =$ the radius of the circular arc, 2α the angle at the center and ϱ the density of the fluid) supplied values differing no more than 22—27 per cent from those experimentally found by Lilienthal. It should be noted, however, that according to an observation of Kutta, Lilienthal's measures, owing to his limited means of experimentation, were not perfectly exact, as he himself agreed.

The hypotheses put forward by Kutta as the basis of his researches were the following: (1) absence of friction; (2) infinite breadth (span) of the lamina; (3) incompressibility of the fluid; (4) absence of vortices at the edges of the lamina. His results were the following: (1) lift different from zero and well determined; (2) direct drag equal to zero.

In the year 1906 the problem was resumed by Joukowski (1847—1921) in Russia, in a note "Sur les Tourbillons Adjoints"[1]. In this note, where the reaction of a body moving through a fluid was made dependent on the circulation of the velocity, Joukowski demonstrated the well known theorem known by his name, applying it to the determination of the thrust on a cylinder exposed to a current directed perpendicular to its axis.

This note written in Russian did not receive sufficient circulation, whereas very great circulation indeed was enjoyed by a subsequent note of the year 1906, written in French, entitled "De la chute dans l'air de corps légers de forme allongée, animés d'un mouvement rotatoire"[2]. The purpose of this note was that "of establishing the theoretical analysis of motion in the case in which a symmetrical body, endowed with an initial rotating velocity, is placed in a current of air directed perpendicular to its axis of rotation".

In this note Joukowski began by pointing out that already in the year 1901 Professor Köppen[3] had described a model of an aeroplane constructed by him, taking as a basis the principle that a body moving with a rotary motion is supported by air. Similarly he called to mind the experiments carried out by Magnus of Berlin to explain why a spherical projectile, if endowed with rotary motion, deviates from its vertical plane, noting that, according to Magnus, this deviating force was due to the fact that if the air molecules, adjacent, for instance, to the point A of the projectile, possess a velocity less than those adjacent, for instance, to the point B, then, on account of Bernoulli's theorem, a force from A toward B ought to arise.

On the value of this force Q, Joukowski added that he had been able in his preceding note of the year 1905 to demonstrate the following general theorem of hydrodynamics:

"If an irrotational two-dimensional fluid current, having at infinity the velocity v, surrounds any closed contour on which the circulation of velocity is $2k$, the force of the aerodynamic pressure acts on this contour in a direction perpendicular to the velocity and has the value

$$Q = 2k\varrho v$$

The direction of this force is found by causing to rotate through a right angle the vector v around its origin in an inverse direction to that of the circulation."

The demonstration of the theorem followed the enunciation.

Now, before proceeding further, let us sum up chronologically the basic publications referring to the lift of the wing of infinite span. These are: a note by Kutta in the year 1902, two notes by Joukowski in the year 1906, and Lanchester's "Aerodynamics" in the year 1907.

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[1] Transactions of the Physical Section of the Imperial Society of the Friends of Natural Sciences, Moscow, vol. XIII, No. 2.

[2] Bulletin de l'Institut Aérodynamique de Koutchino, Fascicule 1, St. Pétersbourg, 1906.

[3] Illustrierte Aeronautische Mitteilungen, No. 1, 1901.

In the year 1910 four new papers an the subject were published: one by Kutta, one by Tchapliguine and two by Joukowski. But before passing to them, a lecture read on September 6th, 1909, at Lausanne by S. Finsterwalder on aerodynamics as the basis of aeronautics[1] should be mentioned, as a clear exposition is there found of the knowledge acquired in aerodynamics up to that time, and the contributions of the three authors, Kutta, Joukowski and Lanchester are examined and commented upon.

We find in this paper the authoritative opinion of a scientist who had watched over the very birth of these researches, since he had been the teacher of Kutta, and Kutta had presented to him in the year 1902 a graduation paper of which his note, already mentioned and published in the same year, was only a part. Rather this publication had been made by Kutta only on the insistence of his teacher who saw clearly the significance of his work.

The lecture by Finsterwalder deserves to be quoted integrally to some extent, owing to its historical importance.

"Recent studies have put in evidence the following fact, that is, that in theoretical hydrodynamics, not only is the case of a uniform ideal motion without resistance perfectly conceivable, but even this motion can be joined to a lift; so that the floating of a heavy body on the air for an unlimited time, without loss of height, does not involve contradiction.

This fact, of difficult agreement with Kirchhoff's results on the motion of a body in perfect fluid, has been recognized only in recent days. Seven years ago Kutta starting from Lilienthal's experiments, constructed the formulae for a current determined by the passage through air of a long and thin lamina, obtained from the surface of a circular cylinder, which moved transversely, maintaining its chord horizontal. As in this motion no surfaces of discontinuity form, no expenditure of energy occurs, whereas at the same time the lamina experiences a thrust perpendicular to the chord of its profile and therefore to the direction of motion. Recently Lanchester has set forth independently of this a number of analogous though less striking examples, revealing again the reason already given by Kutta, why in this case, contrary to the statement of Kirchhoff, no resistance is found. The fact depends on the circumstance that Kirchhoff tacitly admitted that the space occupied by the air is not divided by the body in motion, or, as is mathematically said, it remains simply connected. When, instead, a body of a great length moves perpendicular to the air, the aerial space remains divided in such a way that it can be practically considered as doubly connected, the motions of the air around the profile of the body then proceed much more naturally and a flow of the fluid along the length of the body can be considered as excluded. But these circulatory motions when they combine with the translatory motion, lead always, in virtue of a theorem established by Joukowski, to a thrust normal to the translatory motion.

On the upper surface the circulatory motion increases the translatory one, therefore there is high velocity and consequently low pressure, while on the lower surface the two movements are opposite, therefore there is low velocity with high pressure, with the result of a thrust upward. And since the latter is perpendicular

[1] "Die Aerodynamik als Grundlage der Luftschiffahrt", Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 1—2, 1910.

to the direction of motion, it does not require work and therefore its presence is not in contradiction with the law of the conservation of energy."

At the beginning of the year 1910 a paper in Russian was published by Tchapliguine entitled "On the Pressure Exercised by a Parallel Two-Dimensional Current on a Submerged Body" [1], where the case was studied in which the wing surface, shaped like the arc of a circle considered by Kutta is not placed with its chord horizontal, but at an angle of incidence β different from zero. The result obtained by Tchapliguine was that this lamina did not determine a continuous current without vortices flowing along its profile, as at the encounter with the leading edge of the lamina the fluid velocity would become infinite. To be able to obtain a continuous irrotational current, it was necessary to fit the leading edge of the lamina with a suitable thickening. If this is done he demonstrated that the flow would cause actually a thrust perpendicular to it, determined by the formula

$$P = 4\pi a \sin\frac{\alpha}{2} \sin\left(\frac{\alpha}{2} + \beta\right) \varrho V^2$$

which, when $\beta = 0$ gave again Kutta's formula. Similarly in the first days of the year 1910 a second note of Kutta appeared dedicated to the study of a two-dimensional current in relation to the bases of the flight problem, "Über eine mit den Grundlagen des Flugsproblems in Beziehung stehende zweidimensionale Strömung" [2]. This new note of Kutta reproduced essentially his dissertation of the year 1902, which by new insistence from his teacher, Professor Finsterwalder, he had resumed and after some revision finally published. In this note there is found the same formula of Tchapliguine already mentioned and deduced by both authors independently. But, even more interesting, was it to learn that the general theorem on lift as the product of circulation by velocity and density, enunciated by Joukowski in the year 1906, had been already found by Kutta, although in a somewhat different form, as early as the year 1902, and for which reason this theorem is indicated today by both the names of Kutta and Joukowski.

Toward the end of the year 1910 Joukowski published a note on two aerofoil sections according to the type of a bird's wing, designed by him and which did not cause infinite values of the velocity. This note devoted to the description of the way in which these profiles could be constructed graphically was entitled "Über die Konturen der Tragflächen der Drachenfliegen" [3]. Another note on the same subject, but of a purely geometrical character, was published by Joukowski in

[1] Mathematical Collections of Moscow, No. 28.

[2] "Sitzungsberichte der königlich Bayerischen Akademie der Wissenschaften", paper presented on January 8, 1910.

[3] "Zeitschrift für Flugtechnik und Motorluftschiffahrt", vol. XXII, No. 2, November 26, 1910.

the same year in Russian under the title "Geometrical Investigations of Kutta's Currents"[1]. The former note began by touching briefly on the two publications of Kutta, (whose priority Joukowski recognized in the enunciation of the theorem of lift) with reference further to the note of Tchapliguine, and to the "Aerodynamics" of Lanchester, expressing on the latter the judgement which has been noted above.

Finally in the year 1911 Kutta and Joukowski published a note each: Kutta "On Two-Dimensional Circulation Currents with Aeronautical Applications"[2] and Joukowski "On Antoinette Type Wings"[3]. In these notes both the authors independently of each other found the same formula for lift experienced by airfoils having a double circular arc as skeleton.

In the same year, 1911, the first publication of L. Prandtl was published on the hydrodynamic theory of lift. But before passing to the examination of Prandtl's contribution to the theory of lift, it is desirable to turn back in order to see what had happened during this time to the researches on the theory of resistance, which we left with the works of Kirchhoff and Lord Rayleigh.

It should be pointed out that in all the studies considered up to that time (1910), and even later, the term *resistance* had not yet taken the modern meaning; that is, the particular signification of *drag*; but by the term resistance of air was understood in general what we now call *aerodynamic action* or *resultant aerodynamic force*. Resistance was therefore divided into *useful resistance* or *useful component of resistance* (modern *lift*) and *resistance propre* (modern *drag*).

Now coming back to Kirchhoff and Lord Rayleigh, we have seen how the former, by applying Helmholtz's idea of the formation of surfaces of discontinuity to the sharp edges of an obstacle, explained and calculated in the year 1869 the resistance of a rectilinear lamina, meeting perpendicularly an irrotational current in an incompressible fluid of constant velocity at infinity, while the latter, in the year 1876, extended this method to the case of a lamina exposed to the current under any angle of incidence α, thus determining the corresponding formula of resistance as a function of the same angle α.

This formula is the following:

$$P = \frac{4\pi \sin \alpha}{4 + \sin \alpha} \varrho V^2 S$$

which, however, gave values too low for small angles of incidence.

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[1] Transactions of the Physical Section of the Imperial Society of Friends of Natural Sciences, vol. XV, No. 1.

[2] "Über ebene Zirkulationsströmungen nebst flugtechnischen Anwendungen". Sitzungsber. d. Bayer. d. Wiss., 1911.

[3] Transactions of the Society of Friends of Natural Sciences of Moscow, vol. XIV, No. 2, 1911.

The problem was successively taken into examination in the year 1881 by D. K. Bobyleff in a note in Russian entitled "Note on the Pressure Determined by a Current of Unlimited Breadth Between Two Laminae Forming any Angle with Each Other" [1], a case of particular interest, since it outlines the rudimentary form of a prow, and which was studied also by Gerlach in a note entitled "Some Remarks on Resistance Experienced by a Flat Lamina and a Keel in a Uniform Current" [2].

Finally on January 5, 1890 Joukowski presented a paper entitled "Modification of the Method of Kirchhoff to determine the Two-Dimensional Motion of a Fluid Given a Constant Volecity along an Unknown Streamline" [3]. In this paper Joukowski set forth a method for the determination of the resistance experienced by a profile compounded of any number of rectilinear segments—a problem studied and solved at the same time, but with a different method, also by Michel in a note entitled "On the Theory of Free Streamlines" [4] and later by Love in the year 1892, in a note entitled "On the Theory of Discontinuous Motions in Two Dimensions" [5] and by Réthy in a note of the year 1895, on the form of jets of incompressible frictionless fluids [6].

However, up to that time resistance had been explained and calculated on the hypothesis of the existence of surfaces of discontinuity, only for bodies having sharp edges. But in the year 1901 T. Levi-Civita (born in the year 1873) thought that the formation of these surfaces should be admitted also for bodies of roundish form. In fact in a letter sent by him in the same year to Professor Siacci, and by the latter, owing to its importance, forwarded to the Academy of Lincei, in the Transactions of which it was published under the title "On the Resistance of Fluid Mediums" [7] he expressed the opinion that to explain the resistance of a three-dimensional body in an actual fluid it was not sufficient to refer only to the viscosity of the fluid, saying that it seemed to himself very strange that the hypothesis of the inviscid fluid, which in so many questions corresponds with sufficient approximation to reality, should lead to

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[1] Journal of the Physico-Chemical Russian Society, vol. XIII, p. 63.

[2] "Einige Bemerkungen über den Widerstand, den eine ebene Platte und ein Kiel von einer gleichförmig strömenden Flüssigkeit erfährt", Civilingenieur, vol. XXXI, 1885.

[3] Published in Russian in the same year in vol. XV of the "Mathematical Collections of the Moscow Mathematical Society" and published again in the year 1930 in No. 41 of the "Transcations of the Central Aero-Hydrodynamical Institute of Moscow".

[4] Philosophical Transactions of the Royal Society, Series A, vol. CLXXXI, 1890.

[5] Proceedings of the Cambridge Philosophical Society, vol. VIII, 1892.

[6] "Strahlenformen von incompressiblen reibungslosen Flüssigkeiten", Mathematische Berichte aus Ungarn, vol. XII, 1895.

[7] "Sulla resistenza dei mezzi fluidi", Rendiconti acc. Lincei, serie V., vol. X. semestre 2, 1901.

results in contradiction with experience only in connection with the problem of resistance. Accordingly, Levi-Civita thought that in dealing with this problem some other element must be introduced, apparently unimportant (or, according to his own words, "innocuous") but actually much farther from reality than perfect fluidity.

This element, as he said, is the hypothesis of the continuity of the fluid in the entire space around the body. In fact he demonstrated in his note that without abandoning the hypothesis of the perfect fluid, resistance was completely explained by replacing the hypothesis of continuity by an hypothesis of discontinuity, the characteristics of which are set forth as follows:

(1) The motion of the fluid produced by a body (the motion being steady relative to the body) presents behind the latter, a surface of discontinuity, extending to infinity from a certain curve on the surface of the body.

(2) The comportment of the fluid molecules constituting the wake is what it would be if they were rigidly connected with the body.

(3) The motion of the fluid outside of the wake is irrotational and satisfies the usual conditions to infinity.

Based on these concepts Levi-Civita arrived at a formula for the calculation of resistance, representing an extension to the three-dimensional case of that established by Lord Rayleigh for the indefinite lamina.

This first note of the year 1901 was followed by the well known paper presented by him on November 11, 1906, to the "Mathematical Club of Palermo" under the title "Wakes and Laws of Resistance"[1]. In this paper, after having mentioned the particular cases of rectilinear laminae studied by his predecessors in order to explain the existence of resistance without departing from the domain of perfect fluids, but substituting for the hypothesis of the unconditioned continuity of velocity, that of the wake, he observed that "a remarkable progress can be reached by resuming the question ab initio and analyzing intimately its mathematical nature" adding that "it is possible to determine indeed the general integral of irrotational motions in which there is a wake".

But before beginning his mathematical treatment to determine the general integral, Levi-Civita observed that while expecting to be able to verify practically, by actual applications to ships, the results of his researches, he thought that there were already experimental proofs enough to show the formation of surfaces of discontinuity when a solid moves through a fluid. These experimental proofs to which Levi-Civita referred, came from two communications of Marey, the one entitled "Les mouvements des fluides étudiés par la chronographie"[2] and the

[1] "Scie e leggi di resistenza", Rendiconti del Circolo Matematico di Palermo, 1st semestre, 1907.

[2] Comtes Rendus, Paris, t. CXVI, pp. 913—923, séance du ler Mai, 1893.

other "Changement de Direction"[1] and from a publication of Ahlborn on the mechanism of aerodynamic restistance[2].

Levi-Civita however, observed that the nature of the discontinuity as appeared from these experiments, was more complex indeed than that postulated by himself in the development of his own theory, expressing himself as follows:

"The region in fact constituted by the wake does not appear in reality to be occupied by a fluid 'solidary' with the body, and much less does it extend to infinity. In it moreover, vortical and turbulent motions occur which attenuate gradually (on account of viscosity and other eventual dissipative actions) in proportion as we go far from the body, so that at a certain distance the state of motion of the fluid no longer presents any vestige of discontinuity; rather all perceptible influence of the motion of the solid ceases, the fluid particles appearing motionless."

Levi-Civita, however, thought that "notwithstanding this difference between the hypothetical wake 'solidary' with the body in the perfect fluid and the actual wake, the resistance could not be far different in the two cases".

He said, in fact, that

"at the end of the vortex surface of discontinuity, extending backwards from the solid, elementary vortex rings separate and insinuate themselves into the wake and then are gradually and continually transformed into heat. On account of the steady character of the phenomena, new vortex rings must continually emanate from the first part of the surface of discontinuity, in contact with the body, descending along the surface, in substitution for those which separate at the end. To the formation of these vortices, as is apparent, only fluid particles can contribute which were previously in contact with the fore part of the surface of discontinuity and they must contribute to this formation according to a law dependent on the regime of motion of the surface".

The result is that resistance depends only on the state of motion of the anterior part of the wake, in contact with the body,

"and since this state of motion is perceptibly the same in both cases of the actual and of the hypothetical accompanying wake, the legitimate conclusion follows that the value of resistance supplied by the calculation can be assumed to be closely approximate to the correct value, notwithstanding the substantial differences which may otherwise exist between the theoretical representation and the actual circumstances".

After having thus justified his mathematical investigation, Levi-Civita proceeds by attacking the problem of discontinuous motions—a problem of high analytical difficulty and which, up to that time had been dealt with in a complete way only for particular cases. He considered the problem actually, according to his own words, *ab initio*, but with limitation, owing to the difficulties, to two dimensions, without, however, causing the problem to lose its practical interest.

[1] Comtes Rendus, Paris, t. CXXIII, pp. 1291—1293, séance du 3 Juin, 1901.

[2] Über den Mechanismus des hydrodynamischen Widerstandes. Abhandlung aus dem Gebiete der Naturwissenschaften, Hamburg, Vol. XVII, 1902.

By his treatment he succeeded in determining, as he had proposed, the general integral of the two-dimensional permanent motions of an indefinite fluid around a submerged obstacle, which determination became possible through the suitable choice of a certain arbitrary function ω.

As a complement to his principal paper and in an appendix to the same, a short note was added by Levi-Civita under the title "Resistance of Friction"[1], demonstrating that

"friction cannot be considered as a prevailing cause of direct resistance, for if it were so, direct resistance with low velocities of translation ought to vary very little and not appear, as in fact, it does appear sensibly proportional to the square of such velocities".

Somewhat earlier, Horace Lamb in the third edition of his "Hydrodynamics", Cambridge, 1906[2] in his turn had given a demonstration that viscosity could not be understood as the principal cause of direct resistance.

Levi-Civita's method for wake investigation was applied in the following year by U. Cisotti (born in 1882) in a paper on jets issuing from a receptacle[3]. Cisotti also limited the question to two dimensions, so that his investigation reduced to the study of a rigid profile (section of the recipient containing the liquid) presenting a mouth (orifice) through which the fluid issues.

Cisotti's investigation was in short the general treatment of that problem which we have seen to have been dealt with by Michel, Love and Réthy for particular cases of rigid profiles formed by rectilinear segments.

In the year 1909 U. Cisotti, in a note on the motion of a solid in a canal[4] extended the method of Levi-Civita to the case in which the obstacle of which the resistance had to be calculated is no longer found in an indefinite fluid, but in the fluid of an indefinite *rectilinear canal*, by supposing both obstacle and motion symmetrical as regards the axis of the canal.

In the year 1911 Henry Villat (born in the year 1879) published on this subject a note in two parts entitled "Sur la résistance des fluides"[5]. In its first part entitled "Sur la résistance des fluides limités par une

[1] "Nota sulla resistenza d'attrito", loc. cit.

[2] The first edition of this well-known treatise goes back to the year 1879 under the title "Treatise of the Mathematical Theory of the Motion of Fluids", the last, that is, the fifth edition, is of the year 1924.

[3] "Vene fluenti", Rendiconti del Circolo matematico di Palermo, vol. XXV, No. 2, 1908.

[4] "Sul moto di un solido in un canale", Rendiconti del Circolo Mat. di Palermo, 1909.

[5] "Annales scientifiques de l'Ecole Normale Supérieure", Troisième Série, Tome Vingt huitième, Paris, 1911.

parois fixe indéfinie", he carried out a new extension of the Levi-Civita method to the case in which the fluid was limited by an indefinite *fixed* wall and the obstacle was of any form. In the second part entitled "Sur le mouvement d'un fluide indéfini autour d'un obstacle de forme donnée", he began by observing how in Levi-Civita's method when that arbitrary function ω had been given *a priori*, both the profile of the obstacle corresponding to this function and all the elements of motion, in particular the resistance of the obstacle, were determined, after which he said:

"Now the true practical problem relating to fluid resistance consists in this: the profile of an obstacle having been given in advance, to determine the corresponding motion and in particular its resistance; which means to seek the function ω corresponding to that profile. This, according to Levi-Civita, constitutes a functional problem of the highest type. The method, for which we are indebted to him, allows us up to the present time, to measure only the difficulty of the question without supplying any practical method of solving it."

Having prefaced this, H. Villat demonstrated how it is possible to substitute for the arbitrary function of Levi-Civita another arbitrary one, quite different, and connected in an intimate and evident way with the form of the obstacle.

This arbitrary function could then be determined so as to give the complete solution of the problem for an obstacle having a profile of a form given in advance, and in particular for an obstacle having the form of a ship, which was the form essentially interesting in all researches at that time.

With these studies, all derived from the method of Kirchhoff, were exhausted all researches which attempted to determine resistance on the basis of the hypothesis of discontinuous surfaces.

We can therefore assert that at the end of the first ten years of this century, both the phenomena of drag and of lift were at last explained without abandoning the field of classic hydrodynamics, the former through Helmholtz's idea of surfaces of discontinuity, the latter through Lanchester's and Kutta's idea of circulation; while another new form of drag connected with lift, in the case of wings of finite aspect ratio, had been also revealed by Lanchester: the "induced" drag.

This progress of the theory was emphasized by N. Joukowski in the foreword of his classic treatise, reproducing his lectures of the year 1911—1912, "The Theoretical Bases of Aeronautics" (Moscow, 1912) saying that "the field of hydrodynamic phenomena which can be explored with exact analysis is more and more increasing" and, after having said that it had become possible to explain in a rational way the air reaction on laminae by means of considerations based on the researches of Lord Rayleigh, Kutta and Tchapliguine, stated that "as causes of the resisting force and of the lifting force of aeroplanes, there appear to be the separation of the streams and the formation around the wings, of the so-called circulation of velocities".

The work of Joukowski here mentioned represented the stenographic report of his lectures taken by his pupils Vetchikin and Chenzoff and reviewed by himself. In it, as is said in the preface, Joukowski proposed "to connect the rich experimental material, up to that time collected in aerodynamic laboratories, with the results of theoretical researches, by means of the fundamental equations of hydrodynamics and of the theory of viscous fluids".

In this volume the following subjects were successively dealt with: the fundamental equations of hydrodynamics, the previous theories and experiments on resistence, vortex theory, airfoil theory and the experiments relating thereto, viscosity of air, and the theory and calculation of propellers [1].

Joukowski, in this treatise, gave special attention to the development of the vortex theory, the importance of which in hydrodynamics was attributed by him to the fact that "while vortices hold a very remarkable part in non-permanent motion, they serve also as a basis to explain permanent motion".

A knowledge of vortex theory and of the circulation of velocities was thus a necessary premise to the aerodynamic study of airfoils, a study constituting the essentially original part of his course. In this part Joukowski set forth, in fact, his theory of lift in connection with circulation, which theory as we have seen, was established by him in his note on the "adjoint vortices" of the year 1906, where for the first time he demonstrated that famous theorem, the honor of the discovery of which he then agreed to share with Kutta: a theorem quite general and applicable to any airfoil section, the whole difficulty of which, in application, consisted only in determining the circulation along the section. The theorem having been applied to the case of the cylinder advancing with its axis perpendicular to the direction of motion, the extension from the circular section to others, was then obtained by the method of conformal transformation.

But in these pages we find likewise the results of his first paper and also those of the others already mentioned; that is of the paper on airfoil

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[1] The first Russian edition of the year 1912 was followed by a first French edition in the year 1916, a second Russian edition in the year 1925, and a second French edition in the year 1931.

The first French edition was enriched with a study by Joukowski on the theory of similitude written expressly for it and with the abstracts of two papers of his published after the year 1912, on the vortex wake of Kármán and on the displacement of the center of pressure.

The propeller theory was omitted in this edition and the descriptive part relating to the experiments was reduced. In the second French edition that part of the airfoil theory relating to resistance was also omitted, whereas several notes on the present stage of science regarding various questions were added by Margoulis.

In the second Russian edition, besides the entire contents of the first, there are found abstracts of the successive notes of Joukowski, more complete and numerous than in the French editions.

sections (the special sections then known under his name) of the year 1910, and of his paper on the Antoinette type airfoil sections of the year 1911. An extensive exposition of the experiments carried out on this matter completed, in this volume, the theoretical investigation of his airfoil sections, in connection with which and their aerodynamic characteristics, the comment by one of his most distinguished pupils, S. Drzewiecki, deserves to be quoted:

"From that time (1912) up to a relatively recent time, the numerous airfoil sections calculated by the Joukowski method, have enjoyed a striking preference, owing first of all to their aerodynamic characteristics always being confirmed by laboratory experiments: if today they have been replaced by other types, this fact depends above all on practical exigencies of construction[1]."

Joukowski followed up his investigation of airfoil lift by a short study of airfoil drag, observing in this connection, that the problem of the determination of drag was the most difficult part of airfoil theory.

In dealing with this subject, Joukowski referred to the same earlier papers by himself to which he had referred in the part relating to airfoil lift, and in addition, explicitly also to the first paper by Kutta of the year 1902. In this paper, as we have seen, one of the assumptions made was that of the absence of vortices at the edges of the arched lamina considered by the author. This assumption, however, as Kutta himself observed, ceased to hold good for high values of the angle at the center of the lamina. For instance, if this angle, indicated by 2α surpassed 180^0. Kutta thought that at the leading edge of the lamina, vortices would arise and a surface of discontinuity form; that is, a separation of the fluid would take place. On the contrary, for small values of 2α, and therefore for very flat laminae, Kutta denied that a disturbance arising at the leading edge, in connection with the formation of vortices, would become important.

Starting from these considerations by Kutta, Joukowski assumed that at the leading edge of the lamina a vortical thickening formed, which when fully developed, separated, flowing along with the current, while another formed in the place of the former, to separate also, and so on. But "such vortices can separate also from the trailing edge of the wing and I think", as he said, "that the cause of all frontal resistances is to be found in the separation of these vortices".

The theory of lift and drag of airfoils was followed in Joukowski's treatise, as above noted, by a study on the viscosity of air and by another on the theory and calculation of propellers, with which the volume closed. Omitting the latter, brief note may be taken of the former.

In it, after having written the equations of motion for an incompressible fluid, taking into consideration viscosity, Joukowski showed

[1] Joukowski, N., "Aérodynamique", Préface de M. Drzewiecki à la deuxième édition, Paris, 1931.

that the terms containing the viscosity coefficient would disappear in case of the existence of a potential, in which case the equations took the same form as those for the motion of a perfect fluid. A fact, he said,

"which shows why a movement with the potential of the velocities is so interesting in practice, and why we always speak of movement with a potential of velocities, while for fluids flowing along given airfoil sections very different motions could be supposed".

In fact from the disappearance of such terms in the equations of motion

"it follows that, when a potential of the velocity exists, viscosity exerts no influence either on the motion or on the distribution of pressure which takes place within the fluid.

The influence of viscosity when a potential of the velocity exists, can only appear at the walls of the recipient, where the boundary conditions must be satisfied".

Passing on to consider the comportment of the fluid along the walls, Joukowski observed that

"on the adhesion of the fluid to the walls in its flow, there are very different opinions. Some investigators think that along the walls no motion of the fluid occurs, others suppose instead that the fluid slides along them.

For my part I think (this has not been exactly confirmed by experiment, but it approximates near enough to the reality) that the fluid velocity along the walls is equal to zero, and that it then rapidly increases until it becomes equal to the theoretical velocity obtained by supposing the existence of a potential of the velocities.

The layer of fluid around the walls, lacking a potential of the velocities, and therefore vortical, is very thin . . . . ."

Further experiments confirmed more and more the exactitude of this hypothesis, which experiment up to that time had not as yet established.

Among the investigators referred to by Joukowski (although without mention by name) who maintained the opinion of a zero velocity along the walls, the most important was L. Prandtl, who with his famous paper on the motion of a fluid with very small friction, presented in the year 1904 at the third Congress of Mathematicians at Heidelberg, laid down the bases of the boundary layer theory[1].

Prandtl (born in the year 1875) began his paper by stating that in classic hydrodynamics, while motion of a frictionless fluid was dealt with in an exhaustive manner, for a fluid with friction, although there were differential equations perfectly describing the nature of the phenomenon, we were not in a position, except for one-dimension problems as dealt with by Lord Rayleigh, to solve the equations in cases other than those in which fluid inertia could be neglected. In other words it was not possible to solve the equations of fluid motion for two and three-dimensional problems, including at the same time the influence of both friction and inertia.

[1] "Über Flüssigkeitsbewegung bei sehr kleiner Reibung", Verhandlungen des III. Internationalen Mathematiker-Kongresses, Heidelberg, 1904, Leipzig, 1905.

In fact from the equations of motion put in vectorial form, as he said, it is seen that in fairly slow motion of permanent type or varying only slowly, the inertial term $\varrho\, dv/dt$ becomes small at will, in comparison with the other terms, with the result that the influence of the forces of inertia can be neglected. On the contrary in fairly rapid motions the inertial term becomes much larger, with the result that in comparison with the latter, the term representing the action of friction can be neglected. And owing to the fact that this second case is exactly that of motions which are in general taken into consideration in practice, the rule has been adopted of using the equations of a frictionless fluid only, with this drawback however that results are obtained, as regards resistance, which are in strong disagreement with experience.

Starting from this statement Prandtl proposed to investigate systematically the laws of motion of a fluid in which friction could be supposed very small, so that the latter could be neglected everywhere except in those places where a high velocity gradient occurs. This idea, as Prandtl said, has shown itself very fruitful, since, while it leads to a mathematical formulation which permits a solution of the problem, it supplies on the other hand a very satisfactory agreement with experience.

The most important question in this new treatment of the problem was the comportment of the fluid along the walls of the body. In this connection Prandtl observed that

"a very satisfactory explanation of the physical process in the boundary layer (Grenzschicht) between a fluid and a solid body could be obtained by the hypothesis of an adhesion of the fluid to the walls, that is, by the hypothesis of a zero relative velocity between fluid and wall. If the viscosity was very small and the fluid path along the wall not too long, the fluid velocity ought to resume its normal value at a very short distance from the wall. In the thin transition layer (Übergangsschicht) however, the sharp changes of velocity, even with small coefficient of friction, produce marked results".

A remarkable consequence of this theory from the standpoint of application, was, according to Prandtl, that

"in given cases, in certain points fully determined by external conditions, the fluid flow ought to separate from the wall. That is, there ought to be a layer of fluid which, having been set in rotation by the friction on the wall, insinuates itself into the free fluid, transforming completely the motion of the latter, and therefore playing there the same part as the Helmholtz surfaces of discontinuity (Trennungsflächen)".

Variation in the value of the viscosity could alter the thickness of the vortical layer, but the entire process of the phenomenon would remain unaltered, so that it was possible even to pass to the limit of viscosity approaching zero, without the flow aspect being changed.

As regards the external conditions, above mentioned, determining the separation of the flow, we have to consider, as Prandtl said, that the phenomenon of separation is produced when an increase of pressure is

established along the wall in the direction of the flow. According to this hypothesis, which he asserted to be "very plausible",

"on an increase of pressure, while the free fluid transforms part of its kinetic energy into potential energy, the transition layers instead, having lost a part of their kinetic energy, have no longer a sufficient quantity to enable them to enter a field of higher pressure, and therefore turn aside from it".

From this Prandtl deduced that:

"while dealing with a flow, the latter divides into two parts interacting on each other; on one side we have the 'free fluid', which can be dealt with as if it were frictionless, according to the Helmholtz vortex theorems, and on the other side the transition layers (Übergangsschichten) near the solid walls. The motion of these layers is regulated by the free fluid, but they for their part give to the free motion its characteristic feature by the emission of vortex sheets (Wirbelschichten)".

Prandtl closed his paper by describing the experiments carried out by himself to test this theory.

It may be noted that the term "boundary layer" (Grenzschicht) which has now become so famous, in this first paper by Prandtl is used once only, while the term usually employed is that of "transition layer" (Übergangsschicht). The term boundary layer came into more definite use in a paper by H. Blasius of the year 1908 entitled "Boundary Layers in Fluids with Small Friction"[1].

In this paper Blasius, confining himself to the two-dimensional field, applied Prandtl's theory to the formation of boundary layers and to the determination of the point of separation of the flow.

Blasius justified the use of Prandtl's theory by observing that the vortices which form in running waters behind solid bodies cannot be adequately explained either by the potential or by the Helmholtz theory.

As regards the former, its insufficiency is due to the fact that it leaves out of consideration both the adhesion of the fluid to the walls, and the separation of the flow, which separation has been experimentally established at given points, with a corresponding emission from the boundary layer into the free flow of fluid with a vortical motion. This separation, indeed was reproduced in Helmholtz's theory of jets, by assuming two potential flows to be in juxtaposition along a given line; the one being the jet, the other the fluid at rest. But besides other possible objections and the disagreement of the results with the experimental data, the insufficiency of the Helmholtz theory was apparent from the fact that it did not satisfy the condition of adhesion which is characteristic of viscous fluids.

Therefore the flow separation of the Prandtl-Blasius theory is essentially different from that of the Helmholtz theory.

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[1] "Grenzschichten in Flüssigkeiten mit kleiner Reibung", Zeitschrift für Mathematik und Physik, vol. LVI, 1908, p. 1.

Such a separation however, as Blasius said, could not be expected along a flat lamina, since there the pressure does not vary along the indefinite flat wall parallel to the motion; on the other hand, separation was certain when we deal with curved surfaces, such as cylinders and spheres.

Blasius calculated the flow around a cylinder which moves with accelerated motion starting from rest, and he showed that the flow at its very beginning approaches very closely to that of the potential theory of ideal fluids, but afterwards, with increase of the thickness of the boundary layer, the fluid separates at those points where pressure becomes maximum, that is, at the rear of the body. The point of separation then moves forward till a condition of balance is re-established: this balance however not being durable, since the vortices which form at the rear of the body are continually increasing till they leave the body surface and are dragged along by the flow.

Another contribution by Blasius in this paper was the correction of a numerical coefficient which was given by Prandtl, in a formula for resistance.

The latter, in fact, through a calculation not of an ultimate character, had found for a flat infinitely thin lamina, parallel to the flow, the value of the coefficient 1.1 + . . .; this value was now corrected by Blasius through a more exact determination, giving 1.327.

To the study of Blasius another of E. Boltze in the same year followed dealing in an analogous manner with the case of forms of revolution [1].

In the same year, 1908, there appeared a third note on this subject by K. Hiemenz, entitled "The Boundary Layer on a Circular Cylinder Immersed in a Fluid with Uniform Current" [2] in which for a pressure distribution experimentally determined, the corresponding numerical calculation was carried out with exactness, and the points of separation of the current obtained by the calculation were compared with those obtained experimentally, revealing a satisfactory agreement between them.

This note closed this first series of studies carried out in Göttingen for the completion and application of Prandtl's theory.

During this period, moreover, two synthetic expositions appeared in which these new ideas were illustrated and commented. One of them was contained in the lecture on "Aerodynamics as a Basis of Aeronautics" read by S. Finsterwalder at Lausanne in September, 1909, already mentioned in connection with the problem of lift.

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[1] "Grenzschichten und Rotationskörper in Flüssigkeiten mit kleiner Reibung", Göttingen, 1908, Dissertation.

[2] "Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten Kreiszylinder", Dinglers Polytechn. Journal, Vol. 326, 1911.

The second exposition, in which some mechanical problems of interest to aeronautics were reviewed, was written by Prandtl himself, and was published under the title of "Some Mechanical Relations Important for the Science of Flight", together with Finsterwalder's Lecture, in the first and following numbers of the Zeitschrift für Flugtechnik und Motorluftschiffahrt [1] just then founded. To be noted, however, is the fact that among the problems considered by Prandtl in this report (relative motion, gyroscopic actions, stability, and resistance of air) the problem of lift was lacking. However, as already noted, Prandtl started his publications on this subject only in the following year, 1911.

Finsterwalder, after having set forth the fact that for bodies endowed with sharp edges it was easy to explain a resistance by means of the hypothesis of surfaces of discontinuity, said

"recently Prandtl has made plausible the formation of surfaces of discontinuity also on rounded bodies, on which the stationary current would never be able to attain infinite values of velocity, in this way explaining the presence of a resistance also in such cases".

After having set forth Prandtl's theory on the formation of these surfaces of discontinuity (as Finsterwalder called them, whereas Prandtl used the more exact term of surfaces of separation—Trennungsflächen or Trennungsschichten) which, at the rear of the body bound a region where the fluid is found in irregular motion and the pressure falls below that which it should be if the flow closed perfectly behind the body, with the result of creating a difference of pressure between front and rear, Finsterwalder concluded that, from this difference, the resistance of the air essentially derives. This part of his lecture concluded by stating that, "although this theory appears insecure and incomplete and although few consequences of a quantitative character can be deduced therefrom", it still had practical value on account of its being able to supply useful indications on the diminution of passive resistance. From it, for instance, may be deduced the fact, as he pointed out, that in the case of an airship it is not so much the form of the nose that matters as the form of the tail, the principal cause of resistance being there. A fact, he continued, which explains the favorable effect of the fish shape and the unfavorable one of the empennage of airships, the resistance of which in the Zeppelins must at least amount to one-third of the resistance of the smooth hull.

As regards Prandtl's report, the part relating to the resistance of the air began with a rapid exposition of the fundamentals of the hydrodynamic theory which leads to the result of no resistance:

"a result which at first may surprise, but which is understood, however, if we consider that the work absorbed by the advance of a body into a fluid must be

[1] "Einige für die Flugtechnik wichtige Beziehungen aus der Mechanik", Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos 1—7, 1910.

transformed in some other form of energy which must be found again within the fluid. But as a variation of energy cannot take place in an ideal fluid which moves irrotationally, it necessarily follows that resistance on this hypothesis must be equal to zero. But in actual fluids all bodies experience a resistance, from which it is deduced that the open disagreement between theory and experience, originates from the two hypotheses of irrotational flow and of the lack of friction".

To the abandonment of these two hypotheses, as Prandtl said, correspond respectively two resistances: to the abandonment of the hypothesis of lack of friction, corresponds *friction* or *surface resistance* and to the abandonment of the hypothesis of absence of rotation, corresponds *vortex* or *form resistance*.

A close investigation of these two resistances with their mutual dependence, and a more detailed description of the phenomenon of separation of the flow, constitute the most important parts of this review.

The surface resistance, as Prandtl said, is produced by the friction of the fluid flowing along the body and depends essentially on the extent of its surface; the form resistance instead, is due to the fact that the flow is not irrotational throughout its entire course, but vortices arise in dependence on the form of the body, which produce a wake dragged along by the body.

Surface resistance is unavoidable; form resistance instead can be reduced to a great extent by a suitable form of the body, permitting a close approximation to irrotational flow.

However, both these resistances, as he said, are closely connected with each other.

As regards frictional resistance, he noted that very reliable experiments, in particular on the flow through capillary tubes, had shown that a fluid adheres to the walls, coming to rest when in contact with them, so that any effect of friction between a solid body and a fluid must be derived from the relative sliding of fluid layers.

However, the laws which friction obeys were at that time, as Prandtl noted, but poorly understood, except that friction seemed to be neither proportional to the surface of the body nor to a simple index of the velocity. A possible formula for the friction resistance seemed, in any event, to be the following: $$R_f = K\,b\,l^n\,V^{n+1}$$
where K is a coefficient dependent on the density and viscosity of the medium and eventually on the roughness of the surface, b the breadth, and l the length of the surface, V the velocity, n an exponent between 0.5 and 1.0, dependent on the type of the flow. For usual smooth surfaces and high velocities it seemed that one could take $n = 0.80$—0.85.

As regards form resistance Prandtl began by observing that this resistance also was essentially dependent on the friction, which by retarding the boundary layer gave rise to the formation of vortices which are the immediate cause of resistance. In fact, as he said,

"by observing more closely the motion near the boundary layer we shall see that wherever are found causes accelerating the fluid current along the surface of the body, there the particles also of the boundary layer are accelerated. Where instead, causes for delay exist, as for instance an increase of pressure along the body surface, there the particles of the boundary layer, which on account of friction have lost a part of their vis viva, are constrained to turn back prior to the particles of the free current. A return current is thus produced in the boundary layer with the result that at given points the boundary layer material accumulates and rapidly forming a vortex, insinuates itself into the free fluid. The vortex then flows downstream with the current and leaves on the body a separation of the current from the surface, which separation is now permanent. Behind the surface of separation (Trennungsschicht) the fluid is generally in irregular vortex motion, while at a greater distance from the body this wake breaks down into single vortices".

As we have seen, a fundamental point of Prandtl's argument is here, as in his previous paper, that separation of the flow along the body surface occurs at those points where pressure increases:

His words in this connection are in fact the following:

"The theory of this separation is still but little developed; the only thing which can be established surely is, that separation always occurs at such points where the fluid flows along the wall with a decreasing motion, that is, where the pressure is increasing."

After that, in order to put in evidence the particular feature of this resistance in connection with the body form, to which it is indebted for its name, Prandtl suggests the putting of a body in motion through a fluid, for instance a lamina perpendicular to the direction of the motion, in order to observe what happens. We shall see, he said, that at the first moment irrotational motion is established, which on the front surface of the lamina involves a maximum of pressure at the center (stagnation point) where the velocity is nothing, while the pressure decreases more and more towards the edges: consequently the phenomenon of separation here does not occur.

A quite different aspect presents itself instead on the rear surface of the lamina. The irrotational motion would here likewise develop a central stagnation point in exact correspondence with the one on the anterior face, and with a corresponding maximum value of the pressure. But this movement cannot, in fact, take place because, owing to the increase of pressure from the edges towards the center, vortices arise which alter the whole configuration of the flow, with the result of causing the disappearance of the maximum of pressure at the back of the lamina. From all this the consequence is deduced that "the form resistance is the result of this different comportment at the front and rear part of the body".

After an exposition of experiments on air resistance by Eiffel in Paris [1], of others made by Frank in Hanover [2], of others by R. Knoller in Vienna

[1] "Recherches expérimentales sur la résistance de l'air", Paris, 1907.
[2] Zeitschrift d. Ver. Deutsch. Ing. 1908, p. 1522.

in a note "Laws of the Resistance of Air"[1] and others by O. Lilienthal, Prandtl closed his review with the following conclusions:

"The hydrodynamic theory of the resistance of air can be made to agree quite satisfactorily with the numerous and in part remarkable peculiarities of the laws of the resistance of air discovered in recent experimental investigations. In particular attention should be called to the fact that the source of resistance is not to be found in what happens at the front of the body, but in the vortices which form behind. Therefore the form of the posterior part of the body has in many cases more importance for resistance than the anterior one. By suitable form the resistance of airship hulls can thus be reduced, causing it to approach closely to the theoretical value of zero.

All theories which endeavor to base resistance on what happens anteriorly must therefore lead to wrong results, and must be rejected. Moreover the greater or less turbulence of air is of great importance for the values of resistance."

The conclusions of a practical character on the form of airship hulls which are found in this report of Prandtl, as in the lecture of Finsterwalder, are characteristic also of the history of airship technique, showing the influence of the aerodynamic researches of that time on the evolution of these constructions.

The investigations by Prandtl and his collaborators of Göttingen on the formation of vortices behind bodies in motion were not isolated, because besides the experimental researches carried out since the year 1902 on the motion of fluids by F. Ahlborn in Germany, there were not lacking interesting studies on the particular subject of vortices in Great Britain and in France. Thus in the year 1907 there had been published a paper by A. Mallock entitled "On the Resistance of Air"[2] in which he proposed to investigate the comportment of the air at high and low velocities (above and below that of sound) making an attempt to find an expression which will represent resistance generally. After having said that "if the after part of the body tapers very gradually so that the streamlines follow its contours, any resistance experienced by it is due to surface friction only", whereas "if the hinder part of the body is flat or tapers quickly, the streamlines leave the surface and the body carries behind it a wake made up of a complex system of eddies whose formation requires a continuous expenditure of work", he states that "the resistances considered in this paper are those of bodies which form a wake".

One of the simplest cases, as he said, is that of a plate moving perpendicular to the direction of the motion which, as a two-dimensional problem, had been solved by Kirchhoff and Lord Rayleigh, while the corresponding solution, in three dimensions had not yet been effected.

This two-dimensional solution, however, as A. Mallock said, does not hold good in the actual case: in fact in order that it could be applied

[1] "Gesetze des Luft-Widerstandes", Flug- und Motortechnik, No. 21 and 22, 1909.

[2] Proceedings of the Royal Society of London, Series A, vol. LXXIX, p. 262, 1907.

to the calculation of a long lamina advancing into the air (with velocity less than that of sound) it would be necessary for the space occupied by the wake and bounded by the surface of discontinuity to be filled with a frictionless fluid. Now in the real case the fluid wake is not at the pressure of the fluid at infinity,

"because the eddies formed at the edge are forever wrapping up and drawing out from behind the plane part of the fluid which is found there. In consequence of this, fluid in the wake is not at rest relatively to the plane, but in its central part is flowing towards the plane to make good the loss of fluid abstracted by the eddies at its sides".

Imagining the phenomenon in the three-dimensional case, the eddies should appear in the wake, according to Mallock, either as a spiral, or if the circumstances were such as to favor the formation of the eddies at some point of the body, a current would form in the direction of the axis of the eddy. In the two-dimensional case instead, Mallock stated that "the eddies may be formed symmetrically and simultaneously at the two edges, or alternatively, in which latter case the wake consists of a series of alternate right-handed and left-handed eddies".

As regards the mathematical treatment of the motion of the fluid in the wake, Mallock thought that owing to the great complication of this motion, such treatment was quite impossible, "except in a statistical manner and even for this, data are at present wanting".

One year afterward, the alternate eddies appeared in a communication by H. Bénard to the Académie des Sciences de Paris, entitled "Formation of Gyration Centers Behind an Obstacle in Motion"[1]. Bénard, who at that time was unaware of Mallock's researches, began his note by observing that the periodicity of gyratory motions, which are generated by uniform conditions in laminar flow, had been already put in evidence in the year 1883 by Reynolds, who made some drawings in this connection, and in the year 1907 by Brillouin. Later in a lecture on the experimental mechanics of fluids read before the Sorbonne on November 13, 1929[2] Bénard, touching on the historical precedents of his own researches on alternate vortices, stated that the first clear photographic records of them had been taken by Marey in the year 1901, preceded in their turn only by the aforesaid drawings of Reynolds.

In his communication of the year 1908 Bénard showed some pictures, obtained with a special kinematographic device of his own on vortices formed behind vertical cylinders terminating frontally with a more or less acute dihedral angle.

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[1] "Formation de centres de giration à l'arrière d'un obstacle en mouvement", Comptes Rendus, vol. 147, p. 839, séance du 2 Novembre, 1908.

[2] "La Mécanique expérimentale des fluides", Leçon d'ouverture de l'Institut de Mécanique des Fluides, Revue scientifique 28 Décembre, 1929.

The cylinders were immersed for a length of about 6 cm. and emerged only for some millimeters. For a sufficient velocity below which no formation of vortices took place (this limiting speed increased with the viscosity and decreased with the transverse breadth of the cylinders) the vortices which were generated periodically, separated alternately to the right and to the left of the rear turbulence (le remous d'arrière) which followed the cylinder, and they rapidly reached their ultimate place, so that behind the obstacle a double row of stationary "funnels" formed: those to the right turning to the right and those to the left turning to the left, divided from one another by equal intervals. When velocities of rotation were low (low velocity of the obstacle or high viscosity of the liquid, and in all cases vortices about to disappear) the "funnels" were sensibly figures of revolution, whereas when velocities of rotation were higher (high velocity of the obstacle or low viscosity of the liquid) the "funnels" of the two rows became altered in form.

Bénard's communication terminated with observed data on the value of the spacing of the vortices in connection with the velocities, the breadth of the cylinders and the viscosity of the liquid.

In the first days of the year 1911 Riabuchinsky published an article describing an experiment of his own in order to show the periodicity of the alternate vortices[1].

In September, 1911 Th. von Kármán (born in the year 1881) presented to the Society of Sciences at Göttingen a paper entitled "On the Mechanism of Resistance Experienced in a Fluid by a Body in Motion" in which the resistance was precisely determined in connection with the alternate vortices[2].

At the end of the year 1911 Kármán presented two new papers, one with the same title as the preceding, "On the Mechanism of Resistance Experienced in a Fluid by a Body in Motion"[3] in order to rectify some points as regards exactness of exposition and correctness of calculations, the other with the title "On the Mechanism of Fluid and Air Resistance", consisting of a revised edition of the first paper, taking into account the results of the second one, and the results of experiments carried out with the collaboration of H. Rubach[4].

The fundamental point of the resistance problem, as Kármán stated in this definitive paper, is this:

"to what limiting configuration does the flow of the viscous fluid around a solid body tend, when we pass to the limiting case of a perfect fluid?

[1] "L'Aérophile", vol. XIX, January, 1911.

[2] "Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt", Nachrichten der Kön. Gesellsch. d. Wiss. zu Göttingen, vol. XII, No. 5, p. 509, 1911.

[3] Nachr. der Kön. Gesellsch. d. Wiss. zu Göttingen, vol. XIII, No. 5, p. 547, 1912.

[4] Th. von Kármán und H. Rubach, "Über den Mechanismus des Fluids- und Luft-Widerstandes", Physikalische Zeitschrift, Jan. 15, 1912.

The fact that we obtain in this case a resistance nearly independent of the viscosity coefficient allows us to conjecture that in this limiting case the resistance is determined by flow types such as can occur in a perfect fluid.

It is now certain that neither the so-called 'continuous' potential flow nor the 'discontinuous' potential flow discovered by Kirchhoff and von Helmholtz, can express properly this limiting case".

The former as giving no resistance, the latter, although giving a resistance proportional to the square of the velocity, supplies values of resistance which do not agree with those determined experimentally. But, besides this, the hypothesis of the "dead water", which according to this theory ought to move with the body, is in contradiction to nearly all observations. Furthermore, in the theory of discontinuous potential motion, the suction effect behind the body is totally missing, since it is assumed that in the dead water which extends to a great distance, there is everywhere the same pressure as in the fluid at infinity; whereas, according to recent measurements, in many cases, the suction effect is of first importance for the resistance, and in any case contributes an important part of this latter.

As regards the fact that, in a perfect fluid, the discontinuous potential flow, although hydrodynamically possible, was not realized, the reason was without any doubt, according to Kármán, to be found in the instability of the surfaces of discontinuity, as had already been recognized by von Helmholtz and more deeply investigated by W. Thomson (Lord Kelvin). The latter in fact published in the year 1894 a note "On the Doctrine of Discontinuity of Fluid Motion in Connection with Resistance Against a Solid Body Moving Through a Fluid"[1] precisely to demonstrate that "the doctrine of discontinuity is very far from an approximation to the truth".

Passing on then to the examination of this instability, Kármán stated:

"A surface of discontinuity can be considered indeed, as a vortex sheet, and it can be shown in a quite general way that such a sheet is always unstable. This can also be observed directly"; in fact observation shows that vortex sheets have a tendency to roll themselves up, that is, we see the vortex intensity of the sheet concentrate around certain points among which it was originally diffused.

This observation leads to the question: "can there exist stable arrangements of isolated vortex filaments which can be considered as the final product of decomposed vortex sheets?"

To answer this question, Kármán, limiting himself to the two-dimensional case, took into consideration two parallel rows of rectilinear infinite vortices of equal strength, but of inverse sense of rotation, and disposed at equal distances, and started to investigate mathematically if it would be possible to find an arrangement of these vortices such as

[1] "Nature", vol. I, pp. 524, 549, 573, 597, 1894.

to permit the whole system, while maintaining its configuration unaltered, to move with a uniform translatory motion.

He found, in fact that in reality two arrangements were possible for which the two parallel rows of vortices were permitted to advance with rectilinear uniform motion. In the first arrangement the vortices were disposed opposite one another symmetrically in the two rows; in the second arrangement vortices of one row were placed opposite the middle points of the distance between two successive vortices of the other row. But Kármán by means of his analysis found that only the second arrangement could be stable, and only for a given ratio of the distance h between the rows and the distance l between two successive vortices of each row. This value of h/l was calculated by Kármán as 0.283, which value in the proximity of the body was somewhat higher, that is, 0.36.

Then having indicated by u the advancing velocity of the vortices (somewhat less than the velocity U of the body, so that the vortices pass down-stream with the velocity $U - u$ relative to the body) Kármán established for the velocity a formula, which, after the introduction of the ratio h/l (determined by means of the aforesaid investigation on the stability) became $u = z/(l\sqrt{8})$, where z indicated the strength of the vortices.

From this point Kármán passed on to the formula for the stream function and by means of this formula calculated the streamlines, some of which closed around the vortices, while the others ran between them. On the other hand by means of suitable experiments carried out with the collaboration of Rubach, he put in evidence the actual comportment of the flow behind flat laminae and circular cylinders in motion in the water, obtaining photographic records, in which the characteristic zigzag configuration already photographed by Bénard and which in the meanwhile had been put in evidence behind balloon models in Great Britain[1] and behind obstacles in Germany by G. v. d. Borne in a paper presented to the Meeting of the Representatives of the Science of Flight at Göttingen in November, 1911[2].

As we can understand from the foregoing, the periodic character of the motion in the so-called "vortex wake" had been noticed before Kármán and contemporaneous with him by other investigators, but the phenomenon could not be explained until Kármán had made his analysis of stability, through which the periodic variations appeared as a natural consequence of the instability of the symmetrical flow.

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[1] Technical Report of the Advisory Committee for Aeronautics, 1910—11.

[2] "Über Strömungserscheinungen an Hindernissen", Verhandlungen der Versammlung von Vertretern der Flugwissenschaft am 2. bis 5. November, 1911, zu Göttingen.

In this connection it is very interesting to follow Kármán's description how, in the development of the phenomenon, the stable configuration is established.

"When a body is set in motion from rest (or conversely, the stream is directed to the body) some kind of 'dividing layer' (Trennungsschicht) is first formed, which gradually rolls itself up, at first symmetrically on both sides of the body, till some small disturbance destroys the symmetry, after which the periodic motion starts. The oscillatory motion is then maintained corresponding to the regular formation of vortices turning respectively to the left and to the right[1]."

Finally passing on to the determination of air resistance, which was essentially the purpose of these researches, Kármán, having indicated by d a coefficient dependent on the form of the body, established for the air resistance the following formula:

$$W = \psi_w \varrho \, d \, U^2$$

where the coefficient of the resistance ψ_w was expressed as a function only of the ratios l/d and u/U (quantities determined experimentally) and in the form: $\psi_w = \left[0.799 \frac{u}{U} - 0.323 \left(\frac{u}{U}\right)^2 \frac{l}{d}\right]$

Kármán closed his study by observing that his theoretical investigations ought to be extended in two directions: on one hand from the two-dimensional to the three-dimensional case, a problem which he thought not to be unsurmountable; and on the other hand, from the experimental determination of the ratios l/d and u/U (which are necessary for the calculation of resistance) to their theoretical determination.

This second problem was, according to him, much more difficult than the former, and it

"cannot be solved without an investigation of the process of vortex formation. An apparent contradiction is brought out by the fact that we have used only the theorems established for perfect fluids while in such a fluid (frictionless fluid) no vortices can be formed. This contradiction is explained by the fact that we can everywhere neglect friction, except at the surface of the body. It can be shown that the friction forces tend to zero when the friction coefficient decreases, but the vortex intensity remains finite. If we thus consider the perfect fluid as the limiting case of viscous fluids, then the law of vortex formation must be limited by the condition that only those fluid particles can receive rotation which have been in contact with the surface of the body.

This idea appears first, in a perfectly clear way, in the Prandtl theory of fluids having small friction. The Prandtl theory investigates those phenomena which take place in a layer at the surface of the body, and the way in which the separation of the flow from the surface of the body occurs.

[1] It should be noted that at the Meeting at Göttingen above mentioned (November, 1911) G. Runge suggested the explanation, by the periodic separation of the vortices investigated by Kármán, of the hissing sound emitted by a stick rapidly agitated in the air; and that in the year 1914 this subject was developed by F. Kruger and A. Laut in a paper in which was studied the hissing sound produced by the rapid motion of a stick in the air and by the wind against obstacles, "Theorie der Hiebtöne", Annalen der Physik, vol. 44, p. 801, 1914.

If we could succeed in bringing these investigations on the method of separation of the stream from the wall into relation with the calculation of the stable configuration of vortex sheets formed in any way whatever, as has been explained in the foregoing pages, this would obviously mean great progress. Whether or not this would meet with great difficulties cannot at the present time be stated".

Of the two problems the solution of which was looked for by Kármán, the former was subsequently faced by Joukowski, who, after having dealt with Kármán's theory of resistance in a paper entitled "Vortex Theory of the Frontal Resistance by Professor Kármán"[1], published in the year 1919 a second paper representing the continuation of the preceding and entitled "Vortex Theory of the Frontal Resistance for the Motion of a Fluid in Three Dimensions"[2].

As regards the second problem, it has not yet been solved.

Arriving at this point, before proceeding further, it would be well to stop for a moment in order to sum up what has so far been set forth in the present chapter.

Having begun our survey with Lanchester's work on the problem of lift (1894) we have followed the development of ideas on the same subject by Kutta, Joukowski and Tchapliguine between 1902 and 1911. In the latter year, as noted, Prandtl's publications in this connection started; but before passing on to them, we resumed the examination of the theoretical problem of resistance which had been left with the Kirchhoff and Lord Rayleigh researches (1876). This examination was then carried forward by setting forth first the studies derived from the method of Kirchhoff, successively mentioning Bobyleff, Joukowski, Love, Michel, Réthy, Levi-Civita, Cisotti and Villat, during a period extending from 1881 to 1911. Then, after having taken into account the classical trea ise of Joukowski, Prandtl's boundary layer theory (1904) was set forth together with the work of his first collaborators at Göttingen, among whom Blasius (1908), closing finally with the experimental and theoretical researches on the alternate vortices by Mallock (1907), Bénard (1908) and Kármán (1911).

Let us now pass on to Prandtl and his airfoil theory. The first two publications in which mention is found of Prandtl's researches on this subject are a remark in a paper by O. Föppl on Air Forces on Flat and Curved Plates[3] and a Göttingen report by O. Föppl on the Lift and Drag of an Elevator Placed Behind the Wings[4].

[1] Presented in November, 1913 and published in the Transactions of the Physical Section of the Association of the Friends of Natural Sciences, Moscow, vol. XVII, 1914.

[2] Transactions of the Central Hydro-Aerodynamic Institute of Moscow, No.1, 1919. An abstract of the first note was published in the two French editions of Joukowski's "Aérodynamique" (1916 and 1931) and both the notes are published in the second Russian Edition (1925).

[3] Jahrbuch IV der Motorluftschiff-Studiengesellschaft, pp. 86, 87, Berlin, 1911.

[4] "Auftrieb und Widerstand eines Höhensteuers, das hinter der Tragfläche angeordnet ist", Ninth Report of Göttingen Institute, Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 14, July 29, 1911.

In the first of these, Föppl makes mention of Prandtl's method, based on a series of observations on airfoils with varying aspect ratio but at constant angle of attack, for passing from the conditions for one aspect ratio to those for infinite ratio. This was the first solution of the problem of passing from the conditions for one aspect ratio to those for another, and while later study has shown that the preferable treatment of this problem lies rather through measurements on a series of foils at constant lift coefficient rather than at constant angle of attack, nevertheless this first effective treatment of this important problem deserves notice at this point.

The second Report dealt with experiments pertaining to a general system of researches to investigate the disturbances produced in the neighborhood, by the current around an airplane wing. Föppl after having set forth the experimental results of his research remarks that:

"they agree very closely with the theoretical investigations by Professor Prandtl on the current around an airplane with a finite span wing. Already Lanchester in his work, 'Aerodynamics' (translated into German by C. and A. Runge) indicated that to the two extremities of an airplane wing are attached two vortex ropes (Wirbelzöpfe) which make possible the transition from the flow around the airplane, which occurs nearly according to Kutta's theory, to the flow of the undisturbed fluid at both sides. These two vortex ropes continue the vortex which, according to Kutta's theory, takes place on the lamina.

We are led to admit this owing to the Helmholtz theorem that vortices cannot end in the fluid. At any rate these two vortex ropes have been made visible in the Göttingen Institute by emitting an ammonia cloud into the air. Prandtl's theory is constructed on the consideration of this current in reality existing".

In a footnote Föppl added that: "the theory had been set forth by Professor Prandtl in a lecture 'Aerodynamics and Aeronautics' delivered in the winter semester, 1910—1911, and which will be published in this journal before long".

In fact in the month of November of the same year 1911 at the Meeting of the Representatives of Aeronautical Science in Göttingen, Prandtl contributed a paper entitled "Results and Purposes of the Model Experimental Institute of Göttingen"[1] which was published in February of the succeeding year.

Prandtl, after having recorded in this paper various other studies and researches carried out in the Göttingen Institute, touches briefly on his airfoil theory with the following words, which constitute his first publication on this subject, and therefore deserve to be quoted integrally:

"Another theoretical research relates to the conditions of the current which is formed by the air behind an airplane. The lift generated by the airplane is,

[1] "Ergebnisse und Ziele der Göttinger Modellversuchsanstalt", Verhandlungen der Versammlung von Vertretern der Flugwissenschaft am 3. bis 5. November 1911 zu Göttingen, 1912, republished in Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 3, February 10, 1912.

on account of the principle of action and reaction, necessarily connected with a descending current behind the airplane. Now it seemed very useful to investigate this descending current in all its details. It appears that the descending current is formed by a pair of vortices, the vortex filaments of which start from the airplane wing tips. The distance of the two vortices is equal to the span of the airplane, their strength is equal to the circulation of the current around the airplane and the current in the vicinity of the airplane is fully given by the superposition of the uniform current with that of a vortex consisting of three rectilinear sections.

Considerations of this kind have succeeded very well in the investigation of the influence which an elevator experiences on account of the wing placed before it.

Such a current system produces at the point occupied by the elevator a fully determined descending current, the strength of which depends on the lift of the wing; correspondingly the elevator experiences a negative lift which disappears only by rotating the elevator in a given manner. The observed rotation agrees quite satisfactorily with that calculated.

The same theory supplies, taking into account the variations of the current on the airplane which came from the lateral vortices, a relationship showing the dependence of the airplane lift on the aspect ratio; in particular it gives the possibility of extrapolating the results thus obtained experimentally to the airplane of infinite span wing. From the maximum aspect ratios measured by us (1:9 to that of 1:∞) the lifts increase further in marked degree—by some 30 or 40 per cent. I would add here a remarkable result of this extrapolation, which is, that the results of Kutta's theory of the infinite wing, at least so far as we are dealing with small cambers and small angles of incidence, have been confirmed by these experimental results[1].

Starting from this line of thought we can attack the problem of calculating the surface of an airplane, so that lift is distributed along its span in a determined manner, previously fixed. The experimental trial of these calculations has not yet been made, but it will be in the near future."

In the year 1912, H. Reissner, in an article for an aeronautical Year Book on the recent progress of flight theory[2], after having touched on the previous investigations of Kutta and Joukowski, limited to the two-dimensional field, mentions Prandtl for having "first attempted to determine in an exact manner how the phenomenon is modified by a finite span wing".

With the same kind of investigations forming the subject of the Föppl note belongs a new Report of the Göttingen Institute by Betz, published in the year 1912, and entitled "Lift and Drag of a Wing in the Vicinity of a Horizontal Plane (The Ground)[3].

While putting over for a later note the more detailed theoretical discussion of the subject, Betz here limited himself to showing that a fixed plane wall at a given distance h from the wing can be hydrodynamically replaced by supposing beyond the wall a reflex image of

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[1] See paper by O. Föppl as referred to on p. 371, Fig. 9, p. 72.

[2] "Wissenschaftliche Fortschritte der Flugtechnik, Jahrbuch der Luftfahrt", herausgegeben von A. Vorreiter. II. Jahrgang 1912, München.

[3] "Auftrieb und Widerstand einer Tragfläche in der Nähe einer Horizontal-Ebene (Erdboden)", Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 17, September, 1912, Tenth Report of Göttingen Institute.

the wing, so that the wall becoming a plane of symmetry, can be suppressed without alteration of the flow.

In the year 1913 another Göttingen Report was published by Betz on researches similar from the theoretical point of view, to those of the preceding one.

In this Report entitled "Lift and Drag of a Biplane"[1] there was compared the lift-drag ratio of a biplane with that of a monoplane.

In the same year 1913, the second publication by Prandtl on his airfoil theory appeared in a chapter on "Fluid Motion" (Flüssigkeitsbewegung) for an Encyclopaedia of Natural Sciences[2] in which is given a complete account of the state of knowledge in the subject of hydrodynamics at that moment, including its latest applications to aeronautics. This chapter reprinted under the title "Abriss der Lehre von der Flüssigkeits- und Gasbewegung"[3] was welcomed by Reissner in a review (in Z. F. M., No. 2, 1914) as the first example of treatment of that "*hydrodynamics of reality*" toward which, for almost ten years, the whole movement of ideas had tended.

In the part devoted to aeronautical applications Prandtl begins with a rapid exposition of

"potential motion with circulation around a body which extends from one extremity to the other of the fluid, thus making the space filled by the fluid doubly connected". He then continues, indicating how, "in the case that the body of an airplane is surrounded on all sides by the fluid, the motion with circulation becomes possible only if, from the extremities of the body, vortex filaments issue, the circulation of which agrees with that of the motion around the body. The two vortex filaments form, at some distance from the airplane, a pair of vortices moving downward with the velocity $\Gamma/2\,\pi\,d$, where the distance d represents nearly the airplane wing span".

He then passes on to a similarly rapid exposition of resistance (friction and form resistance) on which recently at the Göttingen Institute he had carried out, with collaboration of G. Fuhrmann, a series of remarkable experiments with models. On this subject two papers deserve to be mentioned, the first by Prandtl of the year 1909 on the plan of these experiments, entitled "The Importance of Experiments with Models for Airships and Airplane Technique and the Installations for these Experiments in Göttingen"[4], the second by Fuhrmann of the year 1912 with the results of these experiments entitled "Theoretical and Experimental Researches on Balloon Models"[5].

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[1] "Auftrieb und Widerstand eines Doppeldeckers", Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 1, 1913. Eleventh Report of Göttingen Institute.

[2] Handwörterbuch der Naturwissenschaften, Jena, 1913.

[3] Jena, 1913, Gustav Fischer.

[4] "Die Bedeutung von Modellversuchen für die Luftschiffahrt und Flugtechnik und die Einrichtungen für solche Versuche in Göttingen", Z. d. Ver. deutscher Ing. 1909. (Lecture at the Annual Meeting of the Society.)

[5] "Theoretische und experimentelle Untersuchungen an Ballonmodellen, Jahrbuch der Motorluftschiff-Studiengesellschaft", Fifth volume, 1911—12, Berlin, 1912.

A result of these experiments was that the form resistance, due to the lack of formation of a stagnation point at the back of the body, owing to the separation of the current, could be very noticeably reduced with a suitable shape of the body, but that also in bodies of the most suitable form (fish-shaped bodies) there always occurred a separation of the current at the tail end.

"Fish-shaped bodies with their back part sharpened are particularly suited to give small resistance . . . . In these bodies of very small resistance the behaviour of pressure stated experimentally agrees very closely with that calculated for potential motion, except at the tail end, where also a separation occurs."

Now from this separation at the trailing edge of an airfoil, according to Prandtl, a vortex originated which was swept away into the current, establishing on the wing an equal and opposed circulation, to which the lift was due.

The passage, in which Prandtl describes the formation of this initial vortex, follows immediately the preceding one, but in smaller print than the rest of the text. Thus:

"The flow with circulation originates from the usual potential flow by the emission of a vortex at the beginning of the movement: this vortex goes farther and farther away along the flow and thereby leaves on the body a circulation, which is equal and opposite to its own."

In a following paragraph entitled "Hydrodynamic Considerations on Fluid Resistance", after having summed up how, in a perfect fluid, owing to the lack of friction, there is no resistance to the motion, while lift can arise, Prandtl takes up again the study of the airplane with a finite span wing, mentioning Lanchester in connection with this subject, and explaining why the actual lift, for a wing span l, maintains itself somewhat inferior to that calculated by the Kutta-Joukowski formula, $L = \varrho \Gamma l V$, for a cylindrical body of very great length l, of translation velocity V and of circulation Γ. The explanation is to be found, according to him, in the two lateral vortices, of which, in a note in parentheses, he touches rapidly on the process of formation from a surface of discontinuity at the trailing edge of the wing. The passage, which is quoted here in its integrity, terminates by stating the presence of a resistance in the wing of finite span.

"The circulatory motion, known also as the perípteral motion (Lanchester) brings with itself in an airfoil surrounded on all sides by the fluid, vortices issuing from the wing extremities. (The vortices develop from a dividing layer which rolls up spirally and which forms at the trailing edge of the wing on account of pressure differences: the fluid flows inward on the suction wing surface and outward on the pressure wing surface).

The pair of vortices has for the length unity the momentum $\varrho \Gamma d$ (d is the distance of vortices between themselves). As the pair of vortices forms every second a new section of Length V, the result is that the momentum to be equated to the lift is $\varrho \Gamma d V$.

By comparing this value with the preceding formula of Joukowski in which l represented the 'lifting length of the wing' we have $d = l$. The pair of vortices react on the form of the current flowing along the wing, determining a descending current and thereby the developed lift diminishes in comparison with that deduced by means of the Kutta-Joukowski calculation for the infinite span wing (Prandtl). Calculations made according to this system for simple wings and for biplanes have found good experimental confirmation.

To this movement is united also a resistance which corresponds to the energy left in the vortex system."

Having arrived at this point of our exposition of Göttingen studies, it will be useful to go back for a moment to Lanchester, whose name we have seen mentioned both by Föppl in the first publication (1911) in which the Prandtl airfoil theory is touched on, and by Prandtl himself in his second publication (1913) on this theory.

This return to Lanchester will allow us to clear up more readily the eventual points, both of contact and of divergence, existing between Lanchester and Prandtl in the development of the finite wing theory.

We remember that in the examination of Lanchester's ideas as set forth by him in the first paragraphs (107—118) of Chapter IV of his "Aerodynamics" (entitled "Motion in the Periptery") it was seen that after having indicated how, in an inviscid fluid, in passing from the case of thé infinite span wing to the finite span, two vortex trunks at the wing extremities must form, he observed that this creation of vortices could not be actually compatible with the concept of the perfect fluid (paragraph 117). To which, however, he added this remark:

"it is worthy of note that the somewhat inexact method of reasoning adopted in the foregoing demonstration seems to be peculiarly adapted, qualitatively speaking, for exploring the behavior of real fluids, though rarely capable of giving quantitative results. The problem in three dimensions will be again examined after reviewing the subject on more rigid lines".

The problem in fact was taken up again by Lanchester at the end of the same chapter (paragraphs 125—127) with an argument, which may be summed up as follows.

In the hypothesis of an airfoil of finite lateral extent, we have to deal with a simply connected space, therefore cyclic motion, characteristic of the airfoil of infinite lateral extent, can no longer exist, and the problem cannot be solved without having recourse to rotational currents, which in short amounts to saying that in a perfect fluid, where vortices cannot form, flying is not conceivable. Now let us take "the case of a viscous fluid and then by supposing the viscosity to become less and less, endeavor to approach the condition of the inviscid". The formation of the two vortices at the wing extremities no longer presents theoretical difficulties, for the core of the vortex trunks can now be formed of a mass of fluid in rotation. These vortices possessed of opposite rotation, by their mutual interaction move through the fluid parallel to one

another in the direction of the motion of the fluid that lies between them, inclining downwards, as fast as they are formed.

If the viscosity of the fluid is small, the dissipation of the vortex motion takes place slowly and the two vortices may persist.

From the ideal fluid conceived as the limiting case of the fluid of very small viscosity, Lanchester passes on to the actual fluid in which, according to him, we are no longer under the same rigid conditions as to the connectivity of the region and we are able to induce vortices with a freedom not possible when viscosity is absent.

These vortices, if we are going to generate them continuously at the right and left hand extremities of the airfoil, can be regarded as forming in effect, taken in conjunction with the airfoil itself, an obstacle to connectivity. In fact, although, like each vortex, they dissipate after a time, we can, however, consider that they persist as long as it is necessary to permit of a cyclic system being established and maintained.

At this point Lanchester remarks that it is probable that these terminal vortices do not each actually consist of a single vortex, but rather of a multiple system of smaller vortices, the formation of which is described by him in the following words, quite analogous to those which, in this connection, were used six years later by Prandtl:

"We may suppose that the air skirting the upper surface of the airfoil has a component motion imparted towards the axis of flight, and that skirting the under surface in the opposite direction, so that when the airfoil has passed, there exists a Helmholtz surface of gyration. This surface of gyration will, owing to viscosity, break up into a number of vortex filaments or vortices after the manner shown" [1].

The Chapter IV of "Aerodynamics" closes by showing how the cyclic motion of the lateral vortices (perpendicular to the translatory motion) superposing on the cyclic motion around the wing (parallel to the translatory motion) prevents the vortex trunks from maintaining themselves parallel to the axis of flight and causes them to spread out.

At the same time the constituent filaments wind around one another like the strands of a rope, forming two vortex cylinders of opposite rotation, which, owing to their mutual action, incline downwards.

Further details on the formation of the surface of discontinuity at the trailing edge of the wing were given by Lanchester in his lecture of the year 1915 entitled "The Flying Machine: the Airfoil in the Light of Theory and Experiment" of which, as noted previously, Prandtl and his collaborators became aware only in the year 1926.

In his reasoning Lanchester starts from the experimental fact that on the wing the pressure difference is maximum in the central region and

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[1] The manner shown in Fig. 83 of his "Aerodynamics" is a figure today reproduced in all expositions of aerodynamics.

decreases toward the extremities, and that this applies both to the positive pressure below and to the negative pressure or suction, above. For the moment, moreover, he does not take into account the mutual influence which these positive and negative pressures may have on each other, but he considers the pressure distribution on each surface separately.

A result of this distribution is that any small unit mass of air passing beneath the airfoil receives an acceleration component toward the airfoil extremities.

Likewise, air passing above the airfoil receives an acceleration inward toward the central region.

When the two adjacent layers of air, one from the upper surface and the other from the lower surface, rejoin at the after edge of the airfoil, they are thus found to have relative motion impressed upon them. Now this is a necessary condition for the formation of a surface of discontinuity . . . . .

Without following Lanchester further in his discussion of the formation of these vortex filaments and their connection with the general vortex system produced by the differences of pressure on the two airfoil surfaces, in which system all the air in the vicinity of the airfoil must take part, we shall limit ourselves here to his demonstration of the existence of the main circulation around the airfoil, deduced from the existence, experimentally established, of the two lateral vortex trunks.

His reasoning is based on the fact that it is impossible that two vortices of "*opposite hand*" should be attached, or attach themselves to any material body without the vortex motion extending continuously around the body from one point of attachment to the other. Thus in the present case, the airfoil forms as it were a bridge connecting the two vortex trunks, constituting with them a single system making the space doubly connected, while the circulation around the foil must have at least the same strength as that existing in the region of attachment. Now, according to Lanchester, all agree in confirming the view, on the one hand that the sudden termination of the vortices at the wing extremities is hydrodynamically impossible, and on the other hand from the analogy with electromagnetic theory, according to which a line of magnetic force passing through a bar of iron is not and cannot be regarded as two separate lines terminating on the surface of the iron, but must rather be considered as continuous and uninterrupted. Lanchester then concludes as follows: "Thus, the author regards the two trailed vortices as a definite proof of the existence of a cyclic component of equal strength in the motion surrounding the airfoil itself."

Now, let us return to the Göttingen studies. On August, 1914 A. Wieselsberger, in a study entitled "Contribution to the Explanation of the V

Flight of Some Migratory Birds"[1], referred again to the "*airplane theory*" set forth by Prandtl in both his publications of the years 1911 and 1913, and "which has already served very well in many researches (see the ninth Göttingen Report)".

On the basis of this theory, Wieselsberger in fact demonstrated that such a disposition is adopted by the birds in order that each of them can utilize the ascending current produced by the wing of its neighbor.

In the same year A. Betz, then an assistant of Prandtl, made mention in a published article of a formula of Prandtl's which is identical with that for the "induced drag", although this term was not in use at that time [2].

Later in the same year Betz published a note entitled "The Mutual Influence of Two Airfoils"[3] in which he took into consideration, from a theoretical point of view, the experimental results set forth in his two preceding notes of the years 1912 and 1914[4].

Betz, after having mentioned Prandtl's publication of the year 1913 and before entering into the subject of his own research, examined Prandtl's simplifying hypotheses adopted in it, which were: 1) That the value of the maximum circulation is equal to that of the average circulation, while in reality the latter is somewhat inferior to the former on account of the lift not being uniformly distributed along the span: 2) That the current around the wing is represented by a rectilinear vortex section prolonging itself in two indefinite lateral vortices: 3) That the two lateral vortices are rectilinear.

The result of his examination was that the first two hypotheses did not produce noticeable errors, while the same could not be said for the third.

After this Betz passed on to deal with his problem, establishing three groups of formulas which gave the variation of the lift and drag of one biplane wing on account of the influence exercised on it by the other wing.

In the same year 1914, precisely one month earlier, Betz had published another note entitled "Researches on Wings with Twisted and Swept Back Tips[5], which has already been mentioned as containing for the

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[1] "Beitrag zur Erklärung des Winkelfluges einiger Zugvögel", Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 15, 1914.

[2] Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos. 16 and 17, p. 239 and Figs. 153—155, 1914.

[3] "Die gegenseitige Beeinflussung zweier Tragflächen", Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos. 18 and 19, 1914.

[4] "Lift and Drag of a Wing in the Vicinity of a Horizontal Plane and Lift and Drag of a Biplane", Tenth and Eleventh Göttingen Reports.

[5] "Untersuchungen von Tragflächen mit verwundenen und nach rückwärts gerichteten Enden", Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos. 16 and 17, 1914.

first time a formula for the calculation of the minimum induced drag for given lift and span together with a graph identical with those found by Lanchester independently in the year 1915.

In the year 1915 Betz published a new note entitled "Investigation of a Joukowski Wing"[1] comparing, for various angles of attack, the values of pressure on an infinite Joukowski wing determined by calculation with those determined experimentally. To realize practically the the two-dimensional problem, Betz placed at the wing extremities two suitable vertical planes or shields.

The experimental results for the distribution of pressure were in good agreement with the theoretical values, except that the actual circulation appeared somewhat inferior to that calculated, on account of friction which had not been taken into consideration.

Betz measured also the lift and drag for various angles of attack. In comparison with the theoretical drag equal to zero, that obtained by measurement was very low over the effective range of the wing, but noticeably higher for too large or too small angles of attack. The actual lift was correspondingly in agreement with the theoretical over the effective range, only everywhere somewhat less. These differences of lift and drag betweeen the theoretical and actual cases are to be explained by the influence of the viscosity of the fluid.

No publications on this subject are found during the year 1916; the years 1917 and 1918 instead were very rich in publications.

Among these there appeared in the year 1917 a note by Betz on the hydrodynamic theory of lift in which he established formulas for the lift and moment for any airfoil sections whatever[2], a research which was continued by him three years later.

Another publication of this year was the volume by R. Grammel on the hydrodynamical fundamentals of flight[3].

But the more important studies on airfoil theory in that period (1917—1918) were published in the confidential Technical Reports of the German military aviation.

During the year 1917 three notes were published in these Reports written respectively, one by Betz and two by Munk. They were: "Influence of the Span and of the Wing Loading on the Air Forces of Airfoils" by A. Betz[4], "Span and Resistance of Air" by M. Munk[5],

[1] "Untersuchung einer Schukouskyschen Tragfläche", Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos. 23, 24, 1915.

[2] "Zur Theorie des Tragflächenauftriebes", Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1917.

[3] "Die hydrodynamischen Grundlagen des Fluges", Vieweg, Braunschweig, 1917.

[4] "Einfluß der Spannweite und Flächenbelastung auf die Luftkräfte von Tragflächen", Technische Berichte, herausgegeben von der Flugzeugmeisterei der Inspektion der Fliegertruppen, Vol. I, p. 98, 1917.

[5] "Spannweite und Luftwiderstand", Technische Berichte, Vol. I, p. 199, 1917.

"Measures on Models of Three Airfoils of Different Span", by M. Munk[1]. All of these notes referred to a problem which had been solved theoretically by Prandtl, viz, given the total air force, the wing span and the density and velocity as well, find that distribution of lift along the span for which the drag is minimum (for drag here is to be understood the induced drag).

Prandtl had found, in fact, that to obtain a minimum of such resistance, the lift must be distributed along the span according to a semi-ellipse, and had, in this case, established formulas sufficient for practice, showing the dependence of the air forces on the aspect ratio, and permitting the calculation, from the values measured for a given aspect ratio, of the corresponding values for a different aspect ratio. Now Betz demonstrates in his note that these formulas could be well applied also to wings having a distribution of the lift different from the semielliptic: deviations could be expected only in the case of wings highly twisted and having a variable section along their span (Tauben wings).

But this note of Betz, in theoretical terms only, had not found among technicians sufficient recognition of the practical importance of its results. This fact led Munk to write his two notes, the first intended to put in evidence and clear up the Prandtl and Betz results, and the other to give an experimental proof of them.

In the first note Munk devoted a considerable part of his discussion to a consideration of the different comportment, in comparison with the usual drag shown by this "*additional drag*" (zusätzlicher Widerstand), which was exactly the same as that which some time later received the name of "*induced drag*". For instance he observed that while the usual resistance increases with the velocity and density of the air, the "additional resistance" diminishes instead. Moreover, he suggests for it the name of "*edge drag*" (Randwiderstand) on account of the fact that

"if the span increases more and more and the influence of the two lateral edges becomes less and less, so does the edge drag become less and less and finally will quite disappear. In an actual airplane the span, in comparison with the wing chord, is never very large and therefore there is always an edge drag".

These three notes, referring to airfoil theory with application to the case of the monoplane, were followed by three other notes in the years 1917 and 1918 referring to the case of the biplane, each respectively the work of Betz, Munk and Prandtl.

The note by Betz, entitled "Calculation of the Air Forces on a Biplane Cellule from the Corresponding Values for Monoplane Wings"[2] consisted in a collection of formulas having their theoretical justification in the Author's paper of the year 1914 (on the mutual influence of two wings).

[1] "Modellmessungen an drei Tragflächen von verschiedener Spannweite", Technische Berichte, Vol. I, p. 203, 1917.

[2] "Berechnung der Luftkräfte auf eine Doppeldeckerzelle aus den entsprechenden Werten für Eindeckertragflächen", Technische Berichte, Vol. I, p. 103, 1917.

The note by Munk was entitled "Contribution to the Aerodynamics of the Lifting Organs of the Airplane"[1] and contained formulas of easy calculation making it possible to pass from a given wing system to any other whatever of the same section; and further to determine the influence of a small variation in a wing system and to calculate from measures executed on a simple wing model the corresponding values for one more complicated; and finally to compare between them the aerodynamic efficiencies of two wing systems. These formulas were deduced from measures on models and the author aimed at replacing by them, the methods, somewhat tedious and not sufficiently exact, used at that time for the aerodynamic calculation of airfoils.

In this note there is found for the first time the expression "*induced drag*" which together with the "*profile drag*" (named by Munk also "*ideal drag*") constituted the "*total drag*".

The new denomination of induced drag was at once accepted by Prandtl, who in his note "Approximate Formulas for the Resistance of Wings"[2] found that the expression "induced drag", suggested by Munk was "*very intuitive*", while the preceding expression "edge drag" corresponded well to the conditions of the monoplane "but did not reproduce the concept of the mutual influence (induction) in the biplane".

Another note by Prandtl, published during the same period, was entitled "Induced Drag of Multiplanes"[3] and contained formulas for the calculation of the induced drag of biplanes and triplanes, assuming the semi-elliptical distribution of lift.

In the month of April, 1918, that is, somewhat before the publication of the note now mentioned, Prandtl had read a paper of a general character before the Fourth Meeting of the Scientific Association for Aeronautics under the title "Lift and Drag of Wings in Theory"[4]. The publication of this paper (which had been considered of a confidential character owing to the state of war) was delayed until the year 1920, in which it appeared[5] with footnotes and appendices, referring to the more recent progress of the science. Later reference will be made to this paper.

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[1] "Beitrag zur Aerodynamik der Flugzeugtragorgane", Technische Berichte, Vol. II, p. 187, 1918.

[2] "Näherungsformel für den Widerstand von Tragwerken", Technische Berichte, Vol. II, p. 275, 1918.

[3] "Der induzierte Widerstand von Mehrdeckern", Technische Berichte, Vol. III, p. 309, 1918.

[4] "Tragflächen-Auftrieb und -Widerstand in der Theorie."

[5] "Jahrbuch der Wissenschaftlichen Gesellschaft für Luftfahrt", pp. 37—65, Berlin, 1920.

Another exposition of a general character, but in more elementary terms, was published at the end of the year 1918 by Betz in an article entitled "Introduction to the Theory of Airplane Wings"[1].

In the same year 1918 also, a study by Th. von Kármán and E. Trefftz must be mentioned, on "Potential Flow Around Given Airfoil Sections"[2] in which the authors extended the Joukowski method for the construction of airfoil sections.

Still earlier, in the year 1913, Trefftz had published a note on the graphical construction of Joukowski airfoils[3] based on a study of O. Blumenthal "On the Pressure Distribution Along Joukowski Airfoils"[4].

Finally to the years 1918—1919 belongs Prandtl's fundamental publication on airfoil theory. Under the name in fact of "Airfoil Theory" Prandtl presented to the Society of Science of Göttingen, in the month of July, 1918, a first paper, completed in the month of December, and published in the Reports of the Society at the end of the same year. This first paper was followed in the month of February, 1919 by another paper under the same title[5].

After having rapidly touched on the precedents of his theory and the work of his collaborators, and having noted that his exposition does not correspond to the historical development of the theory, but is based instead on his most recent views on the matter, Prandtl, in his first paper, deals with the following four subjects: General Principles, General Theory of Steady Motion, Necessary Simplifications, Applications to the Monoplane.

Prandtl begins the first section by observing that, to pass from the problem of the infinite wing in a frictionless fluid (a problem the solution of which for some simple profiles had been known for some time) to the problem of the finite wing it is necessary to have recourse to vortices, as "has been explained with much detail by Lanchester: Aerial Flight, vol. I, paragraphs 124—127". In connection with this

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[1] "Einführung in die Theorie der Flugzeug-Tragflügel", Die Naturwissenschaften, Nos. 38 and 39, 1918.

[2] "Potentialströmung um gegebene Tragflächen-Querschnitte", Zeitschrift für Flugtechnik und Motorluftschiffahrt, p. 111, 1918.

[3] "Graphische Konstruktion Joukowskischer Tragflächen", Zeitschrift für Flugtechnik und Motorluftschiffahrt, No. 10, 1927.

[4] "Über die Druckverteilung längs Joukowskischer Tragflächen", ibidem, 1913.

[5] "Tragflächentheorie", I. Mitteilung, Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Klasse, 1918, II. Mitteilung, ibidem, 1919. These two papers together with the Prandtl paper of the year 1904 on the fluid of small friction, a paper by Betz on "Propellers with Minimum Loss of Energy" and a rich bibliography, were re-published later under the title "Vier Abhandlungen zur Hydrodynamik und Aerodynamik", L. Prandtl und A. Betz, Göttingen, 1927.

author Prandtl adds: "In Lanchester there are found several views very similar to those set forth in the following pages. However, he was lacking in the quantitative elaboration which is a necessary condition for success."

This condition of "having recourse to vortices" having been established Prandtl passes on to explain how the formation of the latter is possible in a frictionless fluid, which "at first seems in contradiction with the Lagrange and Helmholtz theorems. But in all applications relating to interactions between a fluid and solid bodies we must conceive the lack of friction as the result of a passage to the limit, from a very small internal friction".

Having here recalled his paper of the year 1904, Prandtl points out that in the fluid of small friction there is a boundary layer where the transition takes place from the velocity of the solid body to that of the free current. This layer in which rotation is different from zero, leaves the body when the motion has lasted a short time, and insinuates itself into the fluid as if it were a free vortex sheet.

If this boundary layer separates forward of the posterior extremity of the body, as is usual in bluff bodied forms, then behind the boundary layer there will remain a vortex sheet or "*deadwater*", and a noticeable drag arises.

In special cases, as in slender forms terminating at the back more or less sharply, the boundary layer, according to experience, can remain attached to the body up to the posterior extremity [1] leaving it only at the geometric locus where the flow closes again.

This geometric locus is sometimes a point (for instance in a spheroid placed in a flow along the direction of the axis of symmetry) but more often it is a line, a "*line of confluence*". In this latter case the boundary layers from both sides enter into the free fluid united in a vortex sheet. The velocities of each side may coincide, as occurs for instance in the steady two-dimensional wing flow, where, owing to the fact that the boundary layers with the transition to infinitely small friction become infinitely thin, every vestige of vortex sheet disappears and only a potential motion remains. But in general the velocities of the two sides of the vortex sheet are different and remain such even when passing to the limit of vanishing friction.

From this, Prandtl concludes that the existence of a superficial vortex sheet (a discontinuity layer in Helmholtz's sense) behind a body in a

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[1] At this point Prandtl observes in a footnote that according to earlier experience, this has occurred only in turbulent boundary layers and at velocities beyond the critical, referring the reader on this subject to his note on resistance of spheres. (Über den Widerstand von Kugeln, Nachrichten der Kön. Ges. d. Wiss. zu Göttingen, p. 177, 1914).

frictionless fluid is quite possible, and arises moreover always at a line of confluence [1].

Forms in which the flow closes in behind the body without the formation of "deadwater" show very small resistance and therefore have high practical importance in cases in which it is desired to obtain high velocities in a resisting medium. But they are also interesting in a special manner since the absence of a vortex region allows a mathematical analysis of the comportment of the flow.

In such cases the agreement of the theoretical with the experimental results is often quite surprising, as was shown in the paper by Fuhrmann of the year 1912, on theoretical and experimental researches on balloon models.

Now the wings as used in practice all have a trailing edge more or less sharp; that is they have a form intended to avoid, as far as possible, deadwater formation. Moreover it is well known from hydrodynamics that an irrotational current, while passing over a sharp edge gives rise to an infinite velocity, a result, however, which in reality cannot occur. But for the velocity to remain finite, the formation of a vortex sheet is necessary.

In order that the velocity remain finite, it is necessary, in fact, that no boundary streamline (Grenzstromlinie) should form in the fluid a re-entrant angle (einspringende Winkel), but this is possible only if the two parts of the flow on the two sides of the angle unite at the edge; that is, the edge is a line of confluence.

We know, moreover, from Helmholtz that in an inviscid fluid the vortex strength of an element once formed, remains unchanged with time; consequently to satisfy the condition of a finite velocity at the edge, it is necessary for the vortex sheets, which gradually form, to take the value essential for this condition.

Now this condition can be satisfied so long as we are dealing with a wing of smooth profile and a sharp trailing edge and moving with a finite and uniform, or continuously varying velocity; for in such case, the vortex sheets so form themselves at the sharp trailing edge that the velocity there remains finite. For discontinuous forms or motions it

[1] Quite recently A. Betz has published a brief note entitled "*Formation of Vortices in Ideal Fluids and Helmholtz's Vortex Theorem*" (Wirbelbildung in idealen Flüssigkeiten und Helmholtzscher Wirbelsatz, Zeitschrift für Mathematik und Mechanik, vol. X, p. 413, 1930) in which he demonstrates in a very simple manner that the possibility of the formation of surfaces of discontinuity is compatible with the hypothesis of a perfect fluid. The basis of his explanation is that the Helmholtz theorem on the impossibility of the formation of vortices in a perfect fluid holds good for vortex cores of finite section and not for surfaces of discontinuity. Now it is always possible to conceive ideal conditions (one of them was imagined by Felix Klein) in which surfaces of discontinuity (which as is well known are quite equivalent to vortex sheets) are formed in a perfect fluid.

is quite impossible, in a frictionless fluid, to avoid infinite velocities at the sharp edges, notwithstanding the formation of vortex sheets. In such cases, however, the law of the formation of the vortices is such as to reduce to a minimum the development of infinite velocities.

From these considerations Prandtl concludes that in order that a frictionless fluid may be considered as the limiting case of a fluid with small friction, classical hydrodynamics must be supplemented by the following axioms and theorems.

"Axiom I. On the lines of confluence (lines along which two flows, previously divided by the body, unite) vortex sheets can arise.

Axiom II. On sharp and projecting edges of the bodies, infinite velocities do not appear or, at least, appear with the maximum limitation.

Theorem I. Projecting edges of the bodies are, for a flow meeting them transversely, always lines of confluence (and therefore an origin of vortex sheets).

Theorem II. The vortex strength, in each new section of the gradually forming vortex sheet, takes that value which is necessary to maintain the velocity finite at the edge, or to limit, as far as possible, infinite values."

Having in this way enlarged the field of classical hydrodynamics in such way as to make the latter suitable for the study of aerodynamic phenomena, Prandtl observes that in calculating the velocities produced by a vortex system, great difficulties arise, owing to the presence of solid bodies, so long as we do not limit ourselves to the consideration of particularly simple cases. In this connection he mentions the two following studies: "Vortex Motion Behind a Circular Cylinder" by L. Föppl [1] and "On the Motion of Single Vortices in a Flowing Fluid" by M. Lagally [2].

These difficulties, however, can be avoided by imagining the bodies replaced by vortex systems, by the action of which the current external to the bodies remains the same as if the latter were present. Making in this way the region occupied by a body a part of the same fluid, the latter becomes singly connected, and the well-known relations between a vortex field and a field of velocity, valid for a fluid of infinite extension, can be applied.

It is necessary, however, to recognize the fact that nothing would be gained from the mathematical point of view as long as we have to deal with bodies of exactly fixed form, because it would then be necessary to find first the vortex distribution concerned, and this would lead exactly to the same mathematical problem. In very many cases however, it is enough to suppose a suitable vortex distribution and then calculate the form of the body belonging to it. With suitable simplifications in each case there can be obtained a flexibility of methods such as could not be expected from the rigid application of the exact theory.

[1] "Wirbelbewegung hinter einem Kreiszylinder", Sitzungsberichte der Bayer. Akad. d. Wiss., Math.-Phys., 1913.

[2] "Über die Bewegung einzelner Wirbel in einer strömenden Flüssigkeit", ibidem, 1914.

As regards the character of the vortices introduced in the place of the solid bodies, it is necessary to note that they do not obey the Helmholtz theorem on the motion of vortices, but their place is fixed by the condition of replacing the given body. In all the relations, however, they are to be dealt with as real vortices, and in particular, although on special hypotheses, the fundamental equations of hydrodynamics hold good for them. To distinguish these vortices, however, from the real or "free" vortices (freie Wirbeln) Prandtl gave to them the name of "*adherent or bound vortices*" (gebundene Wirbeln).

In the second section "General Theory of Steady Motion" Prandtl begins by deducing the fundamental equation for the easiest and most important case, that is, for the permanent motion of a fluid of infinite extent, homogeneous and incompressible, finding the analogue of the Kutta-Joukowski theorem expressed for unit volume, and moreover establishing the theorem that "the free vortex lines, in the assumed conditions are identical with the streamlines".

Basing himself on the relations established, Prandtl is also in a position to answer the question as to what system of vortices is adapted to substitution for a solid body in a flow, without giving rise to the formation of deadwater. The answer is as follows:

"Let us conceive the interior of the body as replaced by fluid at rest having a pressure $p_0 + q$ equal to that existing at a stagnation point. At the place of the body surface, a vortex sheet then appears with a sudden rise of velocity equal to V. This vortex sheet is the system of adherent vortices sought[1]."

Continuing to apply well known relations of hydrodynamics, Prandtl demonstrates that the free vortices take rise where there is a variation in the strength of the bound vortices, a result expressed vectorially by the relation:

$$div\ \gamma + div\ \varepsilon = 0$$

γ being the bound and ε the free vorticity.

The sources of the vorticity γ (bound vortex system) are found for the most part towards the wing extremities and therefore the principal vortices can be expected behind the said extremities.

Finally from the aforesaid relations, Prandtl deduces the expression for the total air force (gesamte Luftkraft) in a lifting space (tragender Raum): which expression is simplified when the bound vortices are all parallel to one another.

The formation of a vortex system behind the wing, Prandtl observes at this point, leads to the consequence that in the motion of the wing there remains kinetic energy in the fluid, causing a drag which must be overcome by means of an energy expenditure, of which he gives the corresponding expression.

[1] p_0 = pressure of the fluid at infinity, V = velocity of the flow at the point in question, q = dynamic pressure of the undisturbed flow.

In the third chapter "Necessary Simplifications", Prandtl sets forth and examines two simplifying conditions, adapted to make possible the mathematical treatment of such problems. These were:

(1) The condition that the air forces are very small. In consequence of this assumed condition, the relations between the forces and the added velocities become linear in form, with all the mathematical advantages which attach to this form of equation. It likewise results that the added velocities due to the influence of the vortices are small in comparison with the velocity of translation, so that in cases where the sum of the velocities arises in a formula, the former can be neglected. It further results that the vortex ropes leaving the ends of the wing will become approximately rectilinear and parallel to the motion of translation.

(2) The assumption that each wing with its system of vortices may be represented by a single so-called "lifting-line", that is, a vortex line passing through the center of gravity of the longitudinal sections of the vortex distribution and with a vortex strength at each point equal to the total circulation for that section of the wing. In the simplest case, for a straight wing monoplane, this substitution gives a single straight line, perpendicular to the direction of motion, and for this case Prandtl proceeds with the development of formulas for lift and drag.

Consideration is then given to the distribution of vortex strength along the line in order that these simplifying conditions may be applied. It is shown that this requirement will be met if the vortex strength (circulation) at the ends of the line (wing) becomes zero. In other words, this means that the lift at the wing tips must vanish. If it is then assumed that the lift over the greater part of the span is uniform, the reduction to zero may be limited to a small length near the tips. It will thus result that the trailing vortices left behind by the line (wing) will be bunched near the ends and can, therefore, to a first approximation, be represented by a single vortex line trailing from each wing tip.

This gives the well known simple form of equivalent vortex system, comprised of three lines, one transverse, denoting the span and the other two drawn from the ends to the rear toward infinity, and of constant vortex strength throughout.

Prandtl recognizes the fact that the earlier researches by himself and his collaborators referred really to this scheme, and adds that the scheme is always suitable when dealing with the study of the influence exercised by the wing at some distance from the vortex system, as was the case in the researches carried out between 1911 and 1914 by O. Föppl, Betz and Wieselsberger on the interactions of wings. On the contrary, the scheme no longer holds good in the study of the auto-influence of the wing.

Finally in the fourth section of his first paper "Applications to the Monoplane", Prandtl, making use of the preceding considerations, deals with the three following fundamental problems:

(1) "Given the distribution of the lift over the span, together with ϱ and V, to determine the wing form reproducing this lift distribution as well as the drag of the same.

(2) Given the wing form, to determine the lift distribution and the drag.

(3) Given the total lift and the wing span, as well as ϱ and V, to determine that distribution of the lift on the span, for which the drag becomes minimum."

The first two are the inverse of each other, but while the first is relatively easy of solution, the second is more difficult.

More important is the third, the solution of which was obtained by Prandtl in the year 1913, thanks to calculations carried out by E. Pohlhausen. The result obtained by Prandtl showed that the minimum induced drag is given by a semielliptical lift distribution, for which distribution Prandtl deduced formulas making it possible to pass from one given aspect ratio to another. It was then the merit of his collaborators, Betz and Munk, as he said, to demonstrate (in their papers of the year 1917, previously mentioned) that these formulas were applicable also to the case of a lift distribution other than elliptical[1].

In relation with this first Prandtl's paper entitled "Aerofoil Theory", it may be noted that the term "induced drag" does not appear except at the very end, where Prandtl after having said that the drag in a real fluid consists of two parts, "that is of the drag here considered and of a drag produced by the air friction and by the harmful formation of vortices, which for a given profile form depends on the angle of attack only", adds in a note in parenthesis, that for these two parts of the drag there have been introduced the two names of *"induced drag"* and of *"profile drag"*.

Prandtl closes his paper by saying that the success of the theory set forth had led him to establish corresponding relations also for composite wing systems, adding that he would give the results of these and of other investigations in a second paper.

This second paper, of which only a rapid sketch will be given, consists of the following sections: Theory of Multiplanes, Lifting Systems of Minimum Drag, Influence of Walls and Free Boundaries, Conditions at a Great Distance from the Wing.

In the first section Prandtl extends the theory of approximation of the first order to any wing system, setting forth again and completing with considerations of his own, two theorems expounded at that time by Munk in a dissertation under the title "Isoperimetric Problems from Flight Theory", presented in the year 1918 and published in the year 1919[2].

These two theorems in Prandtl's words are:

(1) "If we choose from a lifting system, the elements of which are all contained in a transversal plane, any two groups, the drag which group 1 experiences from the velocity field of group 2 is equal to that of group 2 in the field of group 1.

[1] Easier and more direct demonstrations of this theorem have been given later.

[2] "Isoperimetrische Aufgaben aus der Theorie des Fluges", Göttingen, 1919.

(2) The total drag of any lifting system remains unchanged if the lifting elements, without changing their lifting forces, are displaced in the direction of flight (or otherwise if their stagger is changed)."

In the second section Prandtl takes into consideration the lifting system of minimum drag.

The conditions by which (for a lifting system with a given form of the middle line of the wings) a given total lift is combined with a minimum of induced drag, had been already determined by Munk in his dissertation, by making use of the calculus of variations. In a much more rapid manner and completing it with considerations of his own, the problem is here solved by Prandtl, establishing on this subject the following proposition:

"A ring-shaped lifting system enjoys the lowest minimum value of induced drag of all the lifting systems which do not surpass its boundaries."

In the two following sections Prandtl sets forth and calculates the influence of walls in wind tunnels, and the conditions at a great distance from the wing and in the vortex wake, as well as their connection with lift and drag.

In the year 1920 R. von Mises published the continuation of his note of the year 1917 entitled "On the Theory of Wing Lift"[1].

In this second part of his investigation, von Mises dealt essentially with the problem of profile forms corresponding to given lift conditions.

In the years 1920 and 1921 three notes appeared on the second of these problems relating to the monoplane dealt with by Prandtl (in the fourth section of his first paper "Aerofoil Theory") which consisted in calculating the lift distribution for a given form of wing.

The first of these three notes is a dissertation by Betz entitled "Contributions to the Theory of Aerofoils with Particular Reference to the Simple Rectangular Aerofoil"[2].

The second is a note by R. Fuchs entitled "Contributions to the Prandtl Aerofoil Theory"[3].

The third by Trefftz, is entitled "On the Prandtl Aerofoil Theory"[4].

In the same year 1921 Trefftz published also two abstracts on the same general subject[5].

[1] "Zur Theorie des Tragflächen-Auftriebes", Zeitschrift für Flugtechnik und Motorluftschiffahrt, Nos. 5 and 6, 1920.

[2] "Beiträge zur Tragflügeltheorie, mit besonderer Berücksichtigung des einfachen rechteckigen Flügels", München, 1919, summed up in the Beiheft 2 of the Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1920.

[3] "Beiträge zur Prandtlschen Tragflächentheorie", Zeitschrift für Mathematik und Mechanik, p. 106, 1921.

[4] "Zur Prandtlschen Tragflächentheorie", Mathematische Annalen, Nos. 3 and 4, 1921.

[5] "Zeitschrift für angewandte Mathematik und Mechanik", p. 206 and Innsbrucker Vorträge, p. 34.

It thus appears that at this time there were appearing frequent publications intended to disclose and explain the ideas developed by Prandtl and his collaborators on foil theory. Earlier, mention has been made of the exposition of an elementary character by Betz in the magazine "Naturwissenschaften" at the end of the year 1918, and of the lecture delivered by Prandtl himself in the same year and which was published two years later.

To this lecture is prefixed a note explaining the reason for the delay of the publication, thus giving opportunity to supplement the text with footnotes and an appendix giving information on the results of more recent researches. Then an introduction and three sections follow, in which Prandtl deals with the theory of the finite wing, that of the multiplane and certain applications.

In the text there are found here and there historical indications. Thus, in connection with the two-dimensional problem Prandtl mentions the earliest researches of Kutta and Joukowski and the more recent ones by Kármán and Trefftz (1918), while in connection with the three-dimensional problem, he points out the work of Lanchester (pp. 125—127 of his "Aerodynamics") with the remark however, that "there are lacking therein the following conclusions which for the first time have provided a quantitative estimate".

Before closing the year 1920, Report 28 of the National Advisory Committee for Aeronautics, Washington is to be mentioned, as it contains a study by G. de Bothezat entitled "An Introduction to the Study of the Laws of Air Resistance of Aerofoils" in which among other researches those of Kármán on fluid resistance are set forth in considerable extent, and the name of Lanchester is mentioned in connection with the tip vortices, but no mention is made of the work of Prandtl and his collaborators.

Really Prandtl's aerofoil theory was, at that time, quite unknown outside Germany. The first non-German scientist to call attention to this theory was E. Pistolesi in a lecture read in the month of January 1921 at the Italian Aerotechnical Association (founded the preceding year in Rome) under the title "The Theory of Vortices in Aerodynamics"[1].

A second paper destined to make known the Prandtl aerofoil theory outside Germany, was published in the same year, 1921 in the U.S.A., written by Prandtl himself at the request of the National Advisory Committee for Aeronautics, under the title "Applications of Modern Hydrodynamics to Aeronautics"[2].

After a rapid exposition of the fundamental propositions of the theory of the inviscid fluid there are set forth in this Report the follow-

[1] "La teoria dei vortici in aerodinamica", Atti dell'Associazione Italiana di Aerotecnica, vol. I, 1920—21.

[2] National Advisory Committee for Aeronautics, Report No. 116, 1921.

ing subjects: researches on airship hulls by Fuhrmann, theory of lift (two-dimensional problem), the finite wing, theory of the monoplane, theory of multiplanes, aerofoils in a tube or in a free jet, application of the airfoil theory to the screw propeller. In this paper also, Prandtl gives some historical indications on the work of his forerunners and collaborators, and in particular puts in evidence the work of Lanchester.

Thus on page 25 of his American Report he notes the fundamental idea set forth by Lanchester in his considerations on the infinite wing ("Aerodynamics", pp. 110—118 and pp. 160—161) "that for the production of lift the air mass at any time below the wing must be given an acceleration downward".

Prandtl sums up as follows, Lanchester's discussion on this subject.

"The question he (Lanchester) asks is: what kind of a motion arises if for a short time the air below the wing is accelerated downward, then the wing is moved forward a bit without pressure, then the air is again accelerated, and so on. The span distribution of the accelerations is known for the case of a plane plate, infinitely extended at the sides, accelerated from rest: the pattern of the acceleration direction is given in Fig. 34[1].

It is seen that above and below the plate the acceleration is downward, in front of and behind the plate it is upward opposite to the acceleration of the plate, since the air is escaping from the plate. Lanchester asks now about the velocities which arise from the original uniform velocity relative to the plate owing to the fact that the plate, while it gives rise to the accelerations as shown in Fig. 34, gradually comes nearer the air particle considered, passes by it, and finally moves forward away from it.

The picture of the velocities and streamlines which Lanchester obtained in this way and reproduced in his book was, independently of him, calculated exactly by Kutta. It was reproduced in Fig. 34. It is seen that as the result of the upward acceleration of the flow away from the wing there is an upward velocity in front of the plate, a uniform downward acceleration at the plate itself due to which the upward velocity is changed into a downward one, and finally behind the plate a gradual decrease of the downward velocity on account of the acceleration upward."

As seen, this description of the air current around the infinite wing imagined by Lanchester, is the same as that set forth in the first pages of this third chapter while dealing with his peripteroid system, a description which is here reproduced in Prandtl's own words, to show the perfect agreement of Prandtl with the basic concept of Lanchester on the production of lift.

This concept however, as appears in Lanchester's "Aerodynamics" and as was explicitely declared by him in his lecture "The Flying Machine: the Aerofoil in the Light of Theory and Experiment of the year 1915", is in effect an application and extension of the Newtonian method as

[1] Fig. 34 of Prandtl's report is the reproduction of Fig. 68 of Lanchester's "Aerodynamics", reproducing that wall diagram exhibited by Lanchester in his Lecture of the year 1894, and representing the incurvation of the streamlines on account of the presence of a plate of an "appropriate smoothly curved form, whose leading and trailing edges are conformable to the lines of flow".

applied by Rankine and Froude to the theory of marine propulsion. It rests directly on the third law of motion, and involves that which has been sometimes termed the doctrine of the continuous communication of momentum and of which the fundamental equation declares that the force resulting from the reaction of the fluid is numerically equivalent to the momentum communicated per second to the fluid.

We may recall at this point that at the beginning of the present chapter, note was made of the fact that already in the last century attempts had been made to account for sustentation in flight by application of this principle as applied by Rankine in his theory of propulsion, and consisting in equating sustentation or thrust to the momentum generated downward or backward respectively.

Now it was precisely the work of Lanchester as set forth in his "Aerodynamics" and as he himself said in the aforesaid lecture,

"firstly, that the immediate application of this method does not lead to conclusions which accord with experience, and secondly, that by the aid of certain auxiliary concepts (introduced into the theory as based on a study of vortex motion) it has been found possible to frame a regime in connection with which the Newtonian method is made to give results which are in close agreement with experiment".

We may now return to some further notice of the various publications which, ten years ago, were concerned with the explanation and extension of the Prandtl airfoil theory.

In the year 1921 two other summaries on the matter were published in Germany: one by Prandtl himself, although not limited to the airfoil theory only, entitled "Modern Progress of the Aeronautical Fluid Motion Theory"[1], the other by E. Everling, entitled "The Modern Aerofoil and Propeller Theory"[2].

But the most complete exposition of the Prandtl airfoil theory was first given in the year 1922 by E. Pistolesi in a detailed report entitled "Theory of Vortices Applied to Lifting Systems"[3].

In this same year there was finally published the well known treatise by Fuchs and Hopf[4] which at length supplied a complete course on the subject.

At the end of the year 1922 and in the year 1923 the Prandtl aerofoil theory was divulged respectively in France by M. Roy and in Great Britain by H. Glauert and A. R. Low.

M. Roy published in fact in No. 39 of December 1922 of the "Collection Scientia" a booklet entitled "Théorie des surfaces portantes. La Théorie

[1] "Die neueren Fortschritte der flugtechnischen Strömungslehre", Zeitschrift des Vereins deutscher Ingenieure, No. 37, 1921.

[2] "Die neuere Theorie der Tragflügel und Luftschrauben", ibidem, No. 44, 1921.

[3] "Teoria dei vortici applicata ai sistemi portanti", Rendiconti dell'Istituto Sperimentale Aeronautico, No. 1, February, No. 2, April, 1922.

[4] "Aerodynamik", pps. VIII + 466, Berlin, 1922.

de Prandtl"[1]. This first publication was followed by others on the same subject and by the same author.

The English publications are: "Some Aspects of Modern Aerofoil Theory" by H. Glauert and "The Circulation Theory of Lift, with an Example Worked Out for an Albatross Wing-Profile" by A. R. Low[2].

In the same year, 1923, English scientists verified the Prandtl airfoil theory experimentally, giving their results in the following publications: "Experimental Tests of the Vortex Theory of Aerofoils" by H. Glauert[3], and "The Prediction on the Prandtl Theory of the Lift and Drag for Infinite Span from Measurements on Aerofoils of Finite Span" by A. Fage and H. L. Nixon[4].

In the year 1924 H. M. Martin published a detailed exposition on airfoil theory which he suggested calling by the names of both Lanchester and Prandtl: "The Elements of the Lanchester-Prandtl Theory of Aeroplane Lift and Drag"[5].

Finally in the year 1926 the textbook "The Elements of Aerofoil and Airscrew Theory" by Glauert made Prandtl's aerofoil theory quite familiar to English students.

At that time a keen interest having arisen in England regarding the contribution brought by Lanchester to airfoil theory, the latter was invited by the Royal Aeronautical Society of London to set forth the development of his researches on this subject which he did in the lecture already mentioned under the title "Sustentation in Flight".

In the following year, 1927, on the invitation of the same Society, Prandtl lectured before them, choosing as his subject "The Generation of Vortices in Fluids of Small Viscosity".

This second fundamental objective of his researches, reaching back to his paper on the fluid of small viscosity of the year 1904, has likewise served as the starting point of a modern series of studies on the problems of viscosity and turbulence[6].

These problems however, toward which at the present moment in all countries experimental and theoretical aero-hydrodynamical investigations are converging, are excluded from the present review, which closes with the Lanchester-Prandtl airfoil theory.

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[1] Paris, Gauthier-Villars, p. 131 with illustrations.

[2] First International Air Congress of London, 1923.

[3] Technical Report of the Aeronautical Research Committee, 1923–24, vol. I, Reports and Memoranda, No. 889.

[4] Ibidem, Reports and Memoranda, No. 903.

[5] "Engineering", pp. 1, 35, 100, 169, 258; 1924.

[6] A detailed account of the work carried out in Germany on this subject is to be found in the paper by J. W. Maccoll "Modern Aerodynamical Research in Germany" (The Journal of the Royal Aeronautical Society, No. 236, vol. XXXV, August, 1930.

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